The 4th Homotopy Group of the 3-Sphere in Cubical Agda

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Types 2022

WHAT?

- A computer formalisation of (most of) **Guillaume Brunerie's** PhD thesis in Cubical Agda
- **Synthetic** proof (in HoTT) of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$

Guillaume Brunerie

WHY?

- Brunerie's theorem is to this date one of the most advanced pieces of mathematics developed in HoTT
- Contains small 'gaps' which have made the theorem considered 'unformalisable'

Guillaume Brunerie

HOW?

- Cubical Agda and some trickery (streamlined proofs, new definitions, etc.)

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- Let's start with a brief overview of Brunerie's proof

Chapters 1-3

Chapter 1–3

- Brunerie constructs a map  $: \mathbb{S}^3 \to \mathbb{S}^2$ (the *Brunerie Map*).

$\mathbb{S}^1 * \mathbb{S}^1$

$\mathbb{S}^2$

$\mathbb{S}$ 

$\mathbb{S}^2 \vee \mathbb{S}^2$

Chapters 1–3

-  lives in $\pi_3(\mathbb{S}^2)$
- There is an equivalence $e : \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$.
- Define $\beta : \mathbb{Z}$ by $\beta = e$ ()
- **Main theorem:** We have $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$.

Chapters 1–3

What's needed?

- The James Construction
  - we used a shortcut, but there's also a full formalisation by KANG Rongji
- The Hopf fibration
- The Blakers-Massey Theorem
  - full formalisation by KANG Rongji
- Whitehead products

Not easy, but doable!

What's left?

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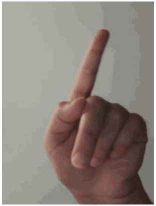


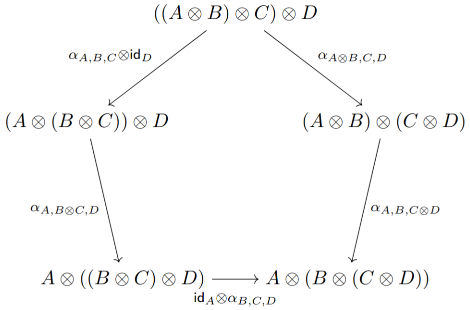
- It's hard!


Chapters 4-6

Chapters 4–6

In order to prove $\beta = \pm 2$, Brunerie introduces a bunch of things:

- Symmetric monoidal structure of smash products



Chapters 4–6

In order to prove $\beta = \pm 2$, Brunerie introduces a bunch of things:

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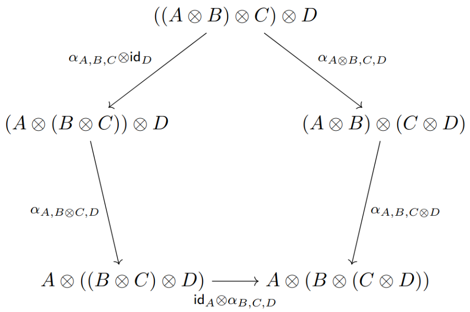
  $\implies$ The graded ring structure of the *cup product* $\smile: H^i(X) \times H^j(X) \to H^{i+j}(X)$



Chapters 4–6

- The *Mayer-Vietoris sequence*

$$\widetilde{H}^{n+1}(D) \xrightarrow{\ i\ } \widetilde{H}^{n+1}(A) \times H^{n+1}(B) \xrightarrow{\ \Delta\ } H^{n+1}(C)$$

$$H^{n}(C) \xrightarrow{\ d\ } \widetilde{H}^{n+1}(D)$$

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Chapters 4–6

- The *Gysin Sequence*

$$\mathbb{S}^{n-1} \longrightarrow E \xrightarrow{p} B$$

$$\dots \longrightarrow H^{i-1}(E) \longrightarrow H^{i-n}(B) \xrightarrow{\smile e} H^i(B) \xrightarrow{p^*} H^i(E) \longrightarrow \dots$$

Chapters 4–6

- The *Hopf Invariant* homomorphism

Definition 5.4.1. Given a pointed map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$, we define

$$C_f := \mathbf{1} \sqcup^{\mathbb{S}^{2n-1}} \mathbb{S}^n,$$
$$\alpha_f := (i^*)^{-1}(\mathbf{c}_n) : H^n(C_f),$$
$$\beta_f := p^*(\mathbf{c}_{2n}) : H^{2n}(C_f),$$

Definition 5.4.2. The *Hopf invariant* of a pointed map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ is the integer $H(f) : \mathbb{Z}$ such that

$$\alpha_f^2 = H(f)\beta_f,$$

where α_f^2 is $\alpha_f \smile \alpha_f$.

Chapters 4–6

- The *Iterated Hopf Construction*

$$
\begin{array}{ccccc}
A & \xleftarrow{\mathsf{fst}} & A \times (A \sqcup^{A \times A} A) & \xrightarrow{(a,x) \mapsto \nu'_a(x)} & \sum_{x : \Sigma A} H(x) \\
\Big\downarrow{\scriptstyle \mathsf{id}} & & \Big\downarrow{\scriptstyle (a,x) \mapsto (a, \nu'_a(x))} & & \Big\downarrow{\scriptstyle \mathsf{id}} \\
A & \xleftarrow[\mathsf{fst}]{} & A \times \sum_{x : \Sigma A} H(x) & \xrightarrow[\mathsf{snd}]{} & \sum_{x : \Sigma A} H(x)
\end{array}
$$

Chapters 4–6

All in all:

- Symmetric monoidal structure of smash products

  $\Longrightarrow$ The graded ring structure of the cup product $\smile : H^i(X) \times H^j(X) \to H^{i+j}(X)$
- The *Mayer-Vietoris* sequence
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Our formalisation:

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Formalisation

- Cubical Agda is excellent because it is entirely constructive
  - Things compute

New definition of cup product
+ Cubical Agda =

- Removes a lot of 'bureaucracy' from certain proofs
- The rest of the formalisation: challenging but straightforward

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- The rest of the formalisation: challenging but straightforward
- Many, many lines of code later, we have it:

$$\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$$

Chapters 4-6?

Teaser: new proof

- In fact, there is a new proof completely bypassing Chapters 4-6.
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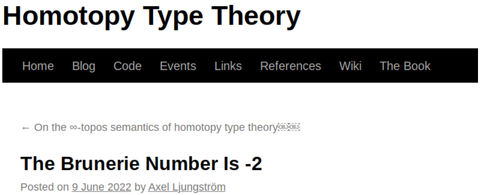
- See my recent post on the HoTT blog for more details.



Homotopy Type Theory

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← On the ∞-topos semantics of homotopy type theory

The Brunerie Number Is -2

Posted on 9 June 2022 by Axel Ljungström

Summary

So we have 3 formalisations:

- A ‘full’ formalisation of Brunerie’s thesis (modulo some trickery)
  - `github.com/agda/cubical/blob/master/Cubical/Homotopy/Group/Pi4S3/Summary.agda`
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Questions?