Cellular Methods in Homotopy Type Theory

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ABSTRACT

In classical mathematics, a CW complex is a topological space which can be built up inductively by gluing together cells of increasing dimension. Due to their good categorical properties, CW complexes have become the main object of interest in algebraic topology. Although their quasi-combinatorial nature suggests that a constructive treatment is possible, there seems to be little literature on the subject – perhaps because of the important role played by the axiom of choice in the classical theory of CW complexes.

In this paper, we present a *synthetic* and *constructive* account of the theory of CW complexes in homotopy type theory. Our first main result is a finitary version of the cellular approximation theorem which, among other things, allows us to construct a cellular homology functor without needing the axiom of choice or relying on a pre-existing notion of homology. Our second main result, which we call the 'Hurewicz approximation theorem', shows that the CW complexes that are n-connected types are precisely the ones that can be presented by a CW structure with no nontrivial cells up to dimension n. This theorem is standard in the classical treatment of CW complexes, but it is far from being obvious in a constructive setting. As a corollary, we give a new proof of the Hurewicz theorem for CW complexes, which relates the first non-vanishing homotopy group of a CW complex with the corresponding homology group. All key theorems presented in this paper have been mechanised in Cubical Agda.

CCS CONCEPTS

• **Theory of computation → Constructive mathematics**; **Type theory**.

KEYWORDS

Homotopy type theory

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1 INTRODUCTION

Homotopy type theory (HoTT) is an extension of intensional type theory which treats types and equalities from a homotopical point of view [19]. In this perspective, types are interpreted as spaces, terms as points, and identity types as paths, with higher identity types encoding higher homotopies. This homotopical interpretation allows us to use type theory as synthetic framework for reasoning about spaces, resulting in proofs that often are simpler and more elegant than the traditional proofs that rely on point-set topology. Indeed, synthetic reasoning has been successfully used to formalise a number of results from algebraic topology, including the computation of various homotopy invariants [6, 2, 9]. In this paper, we develop the theory of CW complexes in HoTT, including cornerstone results such as the cellular approximation theorem, cellular homology, and the Hurewicz theorem.

A remarkable aspect of HoTT is that it is fully constructive by default. While it is possible to postulate classical reasoning principles such as the law of excluded middle or the axiom of choice, we will strive to avoid them in our development in order to remain as general as possible. As a consequence, our results can be interpreted in a wide variety of models – in particular, theorems proved in constructive HoTT are valid in every ∞-topos [13, 17]. Thus, a direct corollary of this paper is that cellular methods are available in the general setting of ∞-toposes.

1.1 Outline and contributions

In Section 2, we outline the basic definitions from HoTT which we will need for our development, with special emphasis on homotopy pushouts and truncations.

In Section 3, we develop our constructive theory of CW complexes. The highlights of this section are a construction of the pushout of two cellular maps (Definition 9 and Proposition 10), and a constructive treatment of the *cellular approximation theorem* for maps and homotopies between CW complexes (Theorems 14 and 21). We also define various categories which, to a varying degree, capture the concept of a cellular space in HoTT, and we study the relations between these categories.

In Section 4, we apply our results to the construction of cellular homology for our categories of cellular spaces. We associate a family of homology groups to every CW complex, and we show that this association is functorial and homotopy invariant using our freshly proved cellular approximation theorems. Note that most proofs and constructions in this section can be interpreted in the setting of cellular *co*homology. Our presentation focuses on homology because, unlike cohomology, there is no treatment of homology in HoTT that is entirely satisfactory as of today.

In Section 5, we prove our *Hurewicz approximation theorem* which shows that CW complexes that are n-connected types are exactly the CW complexes that admit a presentation with no nontrivial cells up to dimension n. (Corollary 41). This theorem is surprisingly powerful, and we can derive the usual Hurewicz theorem (which relates homology groups and homotopy groups of sufficiently connected spaces) as a straightforward corollary.

We should emphasise that our two main contributions (namely, the cellular approximation theorems and the Hurewicz approximation theorem) are statements whose traditional proofs rely on the axiom of choice, and are thus fundamentally non-constructive.

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It was a surprise to us that we could prove them in constructive HoTT. We hope that the subtle constructivity issues will interest also the logician who is not necessarily well-versed in HoTT.

Formalisation. This work is part of a larger programme which aims to develop computational methods for determining the homology, cohomology and homotopy groups of sufficiently well-behaved spaces in HoTT. In particular, several of our results play a crucial role in the recent breakthrough paper on the finite presentability of homotopy groups of spheres using the *Serre finiteness theorem* [1]. The aforementioned programme also aims to implement these methods in `Cubical Agda` in order to obtain formally certified tools that can be used to compute the homotopical invariants of a space from a concrete description. Accordingly, this paper is accompanied by a mechanisation of our results in `Cubical Agda`, which is available at https://github.com/agda/cubical/blob/master/Cubical/Papers/CellularMethods.agda. This development has been incorporated in the mechanisation of the Serre finiteness theorem.

Note that, since the results of this paper are the workhorse that underlies the computational content of the Serre finiteness theorem, it is important that their proofs be computationally efficient. For this reason, we have made a compromise in our formalisation: we have chosen to work with CW complexes which have only a *finite* number of cells in each dimension. In the paper however, we allow for the cells of our CW complex to be indexed by any *projective* set, as we will explain in Definition 5. However, note that all of proofs presented here are sufficiently modular and may be read with either 'finite sets' or the more general 'projective sets' in mind.

1.2 Related work

Our definition of CW complexes is based on the definition given by Buchholtz and Favonia in their work on cellular cohomology [4]. In this paper, we develop the theory quite a bit further: we define cellular maps and cellular homotopies, and we prove their appurtenant approximation theorems. This lets us prove that cellular (co)homology is functorial and homotopy invariant without having to rely on a pre-existing notion of (co)homology.[1] This is especially invaluable for the construction of cellular *homology*, as there is no pre-existing notion of homology that has been developed to the same extent as Eilenberg–MacLane cohomology.

As of today, the most satisfactory treatment of homology in HoTT is given by Graham, who defines it using the Eilenberg–MacLane prespectrum [8]. In [7], Christensen and Scoccola give a proof of the Hurewicz theorem for Graham's definition of homology. Their strategy is fundamentally different from ours: instead of using an approximation theorem, their proof mostly follow from homotopical algebra computations. In any case, we consider that our most significant contribution is the Hurewicz approximation theorem (since it is required for the application to the Serre finiteness theorem), of which the standard Hurewicz theorem is only a pleasant corollary.

[1]Functoriality and homotopy invariance are not proved explicitly by Buchholtz and Favonia, but they follow from their comparison theorem between cellular and Eilenberg–MacLane cohomology.

2 BACKGROUND

In this section, we give a brief introduction to the elementary constructions and facts from HoTT which are used throughout the paper. We assume some level of familiarity with HoTT and refer the reader to the *HoTT Book* [19] whose notation we, for the most part, stay consistent with in this paper. Another excellent introduction is given by Rijke in [16]. The reader who is already too familiar with HoTT may want to skip directly to Section 3.

Π-types. We borrow `Agda`'s notation for dependent products and often write $(a : A) \to B\,a$ instead of $\Pi_{x:a} B\,a$. When the codomain of a Π-types does not depend on its domain, we write it as $A \to B$. We may still use the traditional Π-notation when convenient.

Path types. Given $x, y : A$, we write $x = y$ for their identity type. We refer to elements of this type as *paths between* x *and* y, and we write $\mathsf{refl}_x : x = x$ for the constant path. The *path induction* rule states that dependent functions $((y, p) : \Sigma_{y:A}(x = y)) \to B(y, p)$ are determined by their action on (x, refl_A).

Universes and pointed types. We write Type for the universe of types (at some implicit universe level) and $\mathrm{Type}_\star$ for the universe of pointed types, i.e. the type of pairs $(A, \star_A)$ where $A : \mathrm{Type}$ and $\star_A : A$. For simplicity, we generally write 'A is a pointed type' and leave the basepoint implicit. We always use the notation $\star_A$ for basepoints.

Pointed functions. given two pointed types A and B, the type of pointed functions $A \to_\star B$ is the type of pairs $(f, \star_f)$ where $f : A \to B$ is a function and $\star_f : f\,\star_A = \star_B$. We often simply write $f : A \to_\star B$ and leave $\star_f$ implicit.

Fibres, equivalences and univalence. We write $\mathrm{fib}_f(b)$ for the homotopy fibre of a function $f : A \to B$ over a point $b : B$. That is, $\mathrm{fib}_f(b) := \Sigma_{a:A}(f\,a = b)$. A function $f : A \to B$ whose fibres are contractible (i.e. whose fibres are pointed by a unique point) is called an equivalence, and in that case, we write $f : A \simeq B$ to signify that f is an equivalence from A to B. Equivalences are always invertible, and we will write $f^{-1} : B \simeq A$ for the induced inverse. The identity $\mathrm{id} : A \to A$ is an equivalence; the univalence axiom says precisely that the function $A = B \to A \simeq B$ defined by path induction by sending refl_A to the identity equivalence is itself an equivalence.

The unit type and the empty type. We write $\mathbb{1}$ for the unit type, i.e. the inductive type with one unique constructor $\star_{\mathbb{1}} : \mathbb{1}$, and we write $\perp$ for the empty type.

2.1 Pushouts

Besides inductive types, we will also make heavy use of *higher inductive types* (HITs), which include path constructors in addition to point constructors. One of the arguably most important HITs in HoTT is the *pushout* of a span.

Definition 1 (Pushouts). *Given a span* $Y \xleftarrow{f} X \xrightarrow{g} Z$, *we define its pushout (as indicated in the diagram below) to be the HIT generated by point constructors* $\mathsf{inl} : Y \to Y \sqcup^X Z$ *and* $\mathsf{inr} : Z \to Y \sqcup^X Z$, *as* *well as a higher constructor* push $: (x : X) \to \mathsf{inl}\,(f\,x) = \mathsf{inr}\,(g\,x)$.

$$\begin{array}{ccc} X & \xrightarrow{\;g\;} & Z \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ Y & \longrightarrow & Y \sqcup^X Z \end{array}$$

Given a span S, we may also write $\mathrm{PO}\,S$ for its pushout. We always take $Y \sqcup^X Z$ to be pointed by $\mathsf{inl}\,\star_Y$ (assuming Y is pointed). Pushouts will allow us to define most spaces of interest in this paper, and they will feature prominently in the definition of CW complexes. The following three instances of pushouts are especially important for our purposes.

Cofibres. We define the *cofibre* of a map $f : X \to Y$, denoted C_f, by $C_f := \mathbb{1} \sqcup^X Y$. To stay consistent with the existing literature, we write cfcod instead of $\mathsf{inr} : Y \to C_f$.

Wedge sums. Given an index type I and a dependent family of pointed types $A : I \to \mathsf{Type}_\star$, we define its *wedge sum*, denoted $\bigvee_{i:I}(A\,i)$, to be the cofibre of the obvious map $I \to \Sigma_{i:I}(A\,i)$. When we specifically wish to reason about the binary wedge sum of two pointed types A and B, we may alternatively define these by $A \vee B = A \sqcup^{\mathbb{1}} B$.

Suspensions. We define the *suspension* of a type X, denoted ΣX, by $\Sigma X := \mathbb{1} \sqcup^X \mathbb{1}$. As is standard practice, we use north and south to refer to $\mathsf{inl}\,\star$ and $\mathsf{inr}\,\star$ respectively, and we write merid $: X \to \mathsf{north} = \mathsf{south}$ instead of push. Suspensions allow us to define spheres inductively by setting $\mathbb{S}^{-1} := \bot$, i.e. the empty type, and $\mathbb{S}^n := \Sigma\mathbb{S}^{n-1}$ for $n > -1$.

Two useful lemmas. Let us state two elementary lemmas concerning pushouts which will be useful later. The following lemma is proved using standard pushout-pasting arguments.

Lemma 2. *Let $f : A \to B$ with A and B. We have*

(1) $C_{(\mathsf{cfcod}:B \to C_f)} \simeq \Sigma A$,
(2) $C_f \simeq \Sigma A \vee B$ if B is pointed and f is constant at $\star_B$.

We will also make use of the 3×3-*lemma* – an incredibly useful result which was first introduced in the HoTT literature by Brunerie [2, Lemma 1.8.3], whose notation we also borrow. Let A_{ij} be the following grid of types, doubly indexed by $I = \{0, 2, 4\}$:

$$\begin{array}{ccccc} A_{00} & \xleftarrow{f_{01}} & A_{02} & \xrightarrow{f_{03}} & A_{04} \\ {\scriptstyle f_{10}}\uparrow & & {\scriptstyle f_{12}}\uparrow & & \uparrow{\scriptstyle f_{14}} \\ A_{20} & \xleftarrow[f_{21}]{} & A_{22} & \xrightarrow{f_{23}} & A_{24} \\ {\scriptstyle f_{30}}\downarrow & & \downarrow{\scriptstyle f_{32}} & & \downarrow{\scriptstyle f_{34}} \\ A_{40} & \xleftarrow[f_{41}]{} & A_{42} & \xrightarrow[f_{43}]{} & A_{44} \end{array}$$

Furthermore, assume that the grid is a commutative diagram, meaning that we have four homotopies witnessing that the four small squares commute. Then the 3×3-lemma says that *taking pushouts over rows and then columns is equivalent to taking pushouts over columns and then rows.* Let us unwrap this statement. Let $A_{\bullet i}$ and $A_{i\bullet}$ denote, respectively, the pushout along column i and the pushout along row i – or, more explicitly, define $A_{\bullet i} := A_{0i} \sqcup^{A_{2i}} A_{4i}$ and $A_{i\bullet} := A_{i0} \sqcup^{A_{i2}} A_{i4}$. The first family fits in a span

$$(A_{\bullet i})_{i\in I} \;:=\; (A_{\bullet 0} \xleftarrow{f_{01} \sqcup^{f_{21}} f_{41}} A_{\bullet 2} \xrightarrow{f_{03} \sqcup^{f_{23}} f_{43}} A_{\bullet 4})$$

and likewise, the second family fits in its own span

$$(A_{i\bullet})_{i\in I} \;:=\; (A_{0\bullet} \xleftarrow{f_{10} \sqcup^{f_{12}} f_{14}} A_{2\bullet} \xrightarrow{f_{30} \sqcup^{f_{32}} f_{34}} A_{4\bullet}).$$

Finally, let $A_{\bullet\square} := \mathrm{PO}\,(A_{\bullet i})_{i\in I}$ be the pushout of the first span and let $A_{\square\bullet} := \mathrm{PO}\,(A_{\bullet i})_{i\in I}$ be the pushout of the second.

Lemma 3 (3×3-lemma). $A_{\square\bullet} \simeq A_{\bullet\square}$

2.2 Truncations

We say that a type A is a (-2)-type if it is contractible, (i.e. if it consists of a unique element) and, inductively, we say that A is an $(n+1)$-type if any identity type $x =_A y$ over A is an n-type. We refer to (-1)-types (i.e. types with at most one element) as *propositions* and to 0-types (i.e. types which satisfy UIP) as *sets*.

In HoTT, any type A can be approximated as an n-type by forming its *n-truncation*, denoted $\|A\|_n$. This type is defined as a HIT with a point constructor $|{-}| : A \to \|A\|_n$ and a few additional constructors forcing $\|A\|_n$ to be an n-type. A detailed implementation can be found in [19, Section 7.3] but will not be needed here; all we shall need is the elimination property of the n-truncation, which says that any (possibly dependent) function $f : (x : \|A\|_n) \to B\,x$ is uniquely determined by its action on canonical elements whenever B is a family of n-types. That is, the map $((x : \|A\|_n) \to B\,x) \to ((a : A) \to B\,|a|)$ is an equivalence. The philosophy of the elimination principle is that whenever we are trying to construct an element of a n-type, we may use $\|A\|_n$ and A interchangeably.

Truncations play a crucial role in that they can be used to formulate several important notions and constructions from traditional mathematics in the language of HoTT:

Choice. We say that a set A satisfies choice, or equivalently that A is a *projective set*, if the canonical map $\|(a : A) \to B\,a\|_{-1} \to ((a : A) \to \|B\,a\|_{-1})$ is an equivalence. The statement that every set is projective is a strong form of the axiom of choice, which is not available in constructive HoTT. On the other hand, the set $\mathsf{Fin}(n) := \Sigma_{i:\mathbb{N}}(i < n)$ is constructively projective, meaning that we do not need any axiom to get choice for families indexed over a finite set. We write pSet for the type of all projective sets.

Existence. We define $\exists_{a:A}(B\,a) := \|\Sigma_{a:A}(B\,a)\|_{-1}$ to encode the notion of propositional existence, which is also called mere existence. When this type is inhabited, we say that there *merely* exists an element $a : A$ such that $B\,a$ holds.

Homotopy groups. Given a pointed type A and an integer $n \geq 1$, we define the nth homotopy group of A by $\pi_n(A) := \|\mathbb{S}^n \to_\star A\|_0$. This type turns out to have a group structure, which is abelian for $n \geq 2$. The construction is functorial via post-composition; for a map $f : A \to_\star B$, we write $\pi_n(f) : \pi_n(A) \to \pi_n(B)$ for the functorial action. Additionally, the construction is invariant under n-truncation, meaning that the canonical map $\pi_n(A) \to \pi_n(\|A\|_n)$ is an isomorphism of groups.

Connectedness. We say that a type A is n-connected if $\|A\|_n$ is contractible. A function $f : A \to B$ is said to be n-connected when all of its fibres are. When f is n-connected, it induces an equivalence on truncations $\|A\|_n \simeq \|B\|_n$ (and thus also on π_n).

Another important fact about n-truncations is that they commute with path types, in the sense that for any $x, y : A$, the canonical map $\|x = y\|_n \to |x| =_{\|A\|_{n+1}} |y|$ is an equivalence [19, Theorem 7.3.12]. This principle, together with the elimination principle for $\mathbb{S}^n$, gives rise to the following elementary 'choice principle' for $\mathbb{S}^n$.

Lemma 4. *Given a type indexed over the n-sphere $A : \mathbb{S}^n \to \mathsf{Type}$, we can define a function*

$$\mathsf{choose}_{\mathbb{S}^n} : ((x : \mathbb{S}^n) \to \|A\,x\|_{n-1}) \to \left\|(x : \mathbb{S}^n) \to A\,x\right\|_{-1}$$

3 CW COMPLEXES IN HOTT

A *CW complex* is a space which is constructed by an iterative process of attaching cells: start with a collection of points (0-dimensional cells), then connect some of them using 1-dimensional line segments to obtain a multigraph, then glue a collection of 2-dimensional discs to the multigraph, then 3-dimensional cells, and so on. This iterative construction is captured by the following definition in type theory:

Definition 5. *A projective* ***CW structure*** *is a sequence of types $(X_{-1} \xrightarrow{\iota_{-1}} X_0 \xrightarrow{\iota_0} X_1 \xrightarrow{\iota_1} \dots)$ together with a* ***cardinality*** *function $c^X_{(-)} : \mathbb{N} \to \mathsf{pSet}$ and an accompanying family of* ***attaching maps*** *$\alpha^X_i : \mathbb{S}^i \times c^X_{i+1} \to X_i$ subject to the following two conditions.*

- (A1) *$X_{-1} \simeq \emptyset$,*
- (A2) *for each $i \geq -1$, the square to the right is a pushout.*

$$\begin{array}{ccc} \mathbb{S}^i \times c^X_{i+1} & \xrightarrow{\mathsf{snd}} & c^X_{i+1} \\ {\scriptstyle \alpha^X_i}\downarrow & & \downarrow \\ X_i & \xrightarrow[\iota_i]{} & X_{i+1} \end{array}$$

The cardinality function indicates how many cells should be added at every stage, and the attaching maps α_i explain how the boundary of each individual $(i+1)$-dimensional cell is attached to the i-skeleton X_i. The pushout condition states that X_{i+1} is obtained from X_i by gluing cones along these boundaries, as in the 'hub and spokes' construction from [19, Section 6.7]. Since we will be using them a lot throughout the paper, we introduce special notation for the pushout constructors of the $(i+1)$-th skeleton X_{i+1}: given $x : X_i$, $y : c^X_{i+1}$ and $s : \mathbb{S}^i$, we write

- $\iota_i\, x : X_{i+1}$ (as indicated in the diagram),
- $\mathsf{cell}\, y : X_{i+1}$, and
- $\mathsf{glue}(s, y)$ for the path $\iota_i\,(\alpha^X_i\,(s, y)) = \mathsf{cell}\, y$.

We write CWstr for the type of all CW structures. We will often denote an individual CW structure by $X_* : \mathsf{CWstr}$ and leave ι_*, c^X_* and α^X_* implicit.

Note that we assume our sets of cells to be projective, i.e. we assume that they all satisfy choice. This will let us perform constructions where we have to make an arbitrary choice for each individual cell, which is a very common scenario. In constructive HoTT, projective structures include at least the ones that have a finite number of cells in each dimension, and these are already sufficient to represent many spaces of interest – all the finite CW complexes, of course, but also some infinite-dimensional ones such as the complex projective plane. In presence of the full axiom of choice ($\mathsf{AC}_{\infty,-1}$ in the notation of [19]), all sets are projective and thus, in that setting, our definition works with arbitrary sets of cells.

Definition 6. *Given a CW structure X_*, we define its* ***sequential colimit*** *to be the HIT X_∞ which consists of*

- *for every $x : X_n$, a point $[x]_n : X_\infty$,*
- *for every $x : X_n$, a path $\mathsf{push}\, x : [x]_n = [\iota_n\, x]_{n+1}$.*

We sometimes write $\iota_\infty\, x$ for $[x]_n$ when n is clear from context.

Definition 7. *We say that a type A is a* ***CW complex*** *if there merely exists some CW structure X_* such that A is the sequential colimit of X_*. Formally, we define*

$$\mathsf{CW} :\equiv \Sigma(A : \mathsf{Type})\,.\,\exists(X_* : \mathsf{CWstr})\,.\,X_\infty \simeq A.$$

In the rest of this paper, we will develop the theory of CW complexes, building up to a definition of cellular homology and a proof of the Hurewicz theorem. In doing so, we will take advantage of the fact that every CW complex is presented by a CW structure, which allows us to construct most properties and objects by induction on the dimension. Thus, our first endeavour shall be the development of a working theory of CW structures, starting with their natural notion of maps.

Definition 8. *A cellular map from X_* to Y_* is a pair (f_*, h_*) where $f_i : X_i \to Y_i$ and $h_i : (x : X_i) \to f_{i+1}(\iota_i\, x) = \iota_i(f_i\, x)$, as depicted on the diagram below:*

$$\begin{array}{ccccccc} X_{-1} & \longrightarrow & X_0 & \longrightarrow & X_1 & \longrightarrow & \dots \\ {\scriptstyle f_{-1}}\downarrow & {\scriptstyle h_0} & \downarrow{\scriptstyle f_0} & {\scriptstyle h_1} & \downarrow{\scriptstyle f_1} & & \\ Y_{-1} & \longrightarrow & Y_0 & \longrightarrow & Y_1 & \longrightarrow & \dots \end{array}$$

In simpler terms, a cellular map is a map which respects the dimensions, in the sense that it sends the n-dimensional skeleton of the domain to the n-dimensional skeleton of the codomain. For simplicity, we generally write $f_* : X_* \to Y_*$ for a cellular map, leaving h_* implicit. Every cellular map from X_* to Y_* gives rise to a function f_∞ between their colimits:

$$\begin{array}{lll} f_\infty : X_\infty \to Y_\infty & & \\ f_\infty\, [x]_n & :\equiv & [f_n\, x]_n \\ \mathsf{ap}_{f_\infty}(\mathsf{push}\, x) & :\equiv & \mathsf{push}\,(f_n\, x) \cdot \mathsf{ap}_{[-]_{n+1}}\,(h_n\, x). \end{array}$$

The identity can be presented as a cellular map, and the obvious composition of two cellular maps yields the composition of the colimits. Furthermore, this composition operation is associative and unital. All this data assembles into a category[2] CWstr whose objects are CW structures, and whose mapping sets are given by (the set truncation of) cellular maps. The colimit operation then defines a functor from CWstr to the category of CW complexes and set-truncated ordinary maps, which we denote by Ho(CW).[3]

The interplay between Ho(CW) and CWstr will be a recurring theme of this paper: our main object of interest is the category Ho(CW), but we find that it does not offer sufficient control over the objects and the morphisms. Instead, we define all of our constructions in CWstr, taking advantage of the inductive description of spaces and maps, before transporting them to Ho(CW). Our

[2] Here, we depart from the terminology of the HoTT book: we use the word 'category' for what they would call a precategory, and we use 'univalent category' for what they would call a category.

[3] We reserve the name CW for the wild ∞-category of CW complexes, which we will not use in this paper; all of our categories are 1-categories.

main tool for this transport step will be the *cellular approximation theorem*, which provides a partial inverse to the colimit functor.

3.1 Pushouts of CW structures

Before embarking on the proof of the cellular approximation theorem, it might be good to look at a concrete example of a CW structure, to help the reader build intuition. For this purpose, let us explain how to construct the homotopy pushout of two cellular maps. This construction will play an important role later down the line, as the definition of a homology theory requires our category of CW complexes to be equipped with pushouts.

Definition 9. *Let X_*, Y_*, Z_* be three CW structures, and let $(f_*, h_*) : X_* \to Y_*$ and $(g_*, k_*) : X_* \to Z_*$ be two cellular maps. We define the pushout of the span $Y_* \xleftarrow{f_*} X_* \xrightarrow{g_*} Z_*$ to be the CW structure $(Y \sqcup^X Z)_*$ defined by letting $(Y \sqcup^X Z)_i$ be the pushout $Y_i \sqcup^{X_{i-1}} Z_i$, i.e. the pushout of the span $Y_i \xleftarrow{\iota_{i-1} \circ f_{i-1}} X_{i-1} \xrightarrow{\iota_{i-1} \circ g_{i-1}} Z_i$.*
Inclusions: *The inclusions $(Y \sqcup^X Z)_i \to (Y \sqcup^X Z)_{i+1}$ are the obvious maps that are induced by the corresponding inclusions for X_*, Y_* and Z_*.*
Cells: *We define the cell cardinalities $c_*^{Y \sqcup^X Z}$ in terms of those of X_*, Y_* and Z_* by letting $c_i^{Y \sqcup^X Z} = c_i^Y + c_i^Z + c_{i-1}^X$.*
Attaching maps: *Finally, we define the attaching maps as*

$$\alpha_i^{Y \sqcup^X Z} \;:\; \textstyle\sum_{c \in \{c_{i+1}^Y, c_{i+1}^Z, c_i^X\}} \mathbb{S}^i \times c \;\to\; Y_i \sqcup^{X_{i-1}} Z_i$$
$$\alpha_i^{Y \sqcup^X Z} := \upsilon_i + \zeta_i + \chi_i$$

where we can define $\upsilon_i := \mathsf{inl} \circ \alpha_i^Y$ and $\zeta_i := \mathsf{inr} \circ \alpha_i^Z$, and then $\chi_i : \mathbb{S}^i \times c_i^X \to P_i$ is defined by $\mathbb{S}^i$-induction: the images of the point constructors are defined as $\chi_i(\mathsf{north}, y) := \mathsf{inl}\,(f_{i+1}(\mathsf{cell}\,y))$ and $\chi_i(\mathsf{south}, y) := \mathsf{inr}\,(g_{i+1}(\mathsf{cell}\,y))$, and the image of the path constructor is defined by letting $\mathsf{ap}_{\chi_i(-,y)}(\mathsf{merid}\ x)$ be the composite

$$\mathsf{inl}\,(\dots) \xrightarrow{\mathsf{ap}_{\mathsf{inl}}\, l} \mathsf{inl}\,(\dots) \xrightarrow{\mathsf{push}\,(\alpha_{i-1}^X(x,y))} \mathsf{inr}\,(\dots) \xrightarrow{\mathsf{ap}_{\mathsf{inr}}\, r^{-1}} \mathsf{inr}\,(\dots)$$

where l has type $f_i(\mathsf{cell}\,y) = \iota_{i-1}(f_{i-1}(\alpha_{i-1}^X(x,y)))$ and is defined by $l := \mathsf{ap}_{f_i}(\mathsf{glue}(x,y)^{-1}) \cdot h_{i-1}(\alpha_{i-1}^X(x,y))$, and similarly for r, which has type $g_i(\mathsf{cell}\,y) = \iota_{i-1}(g_{i-1}(\alpha_{i-1}^X(x,y)))$.

Proposition 10. *Definition 9 satisfies (A1) and (A2).*

Proof. The proof mostly follows from the 3×3 lemma. The interested reader can consult the formalised version. ◻

Note that the colimit of this definition is the expected pushout:

$$\operatorname*{colim}_{i\to\infty}\Big(Y \sqcup^X Z\Big)_i = \operatorname*{colim}_{i\to\infty}\Big(Y_i \sqcup^{X_{i-1}} Z_i\Big) \simeq Y_\infty \sqcup^{X_\infty} Z_\infty$$

and thus Definition 9 does indeed provide a CW structure for the pushout of a span in CWstr. Now, if we want to extend this construction to the category Ho(CW), we need to work with arbitrary maps between the colimits instead of cellular maps. This is one out of a handful places where cellular approximation is needed.

3.2 Finite structures and the cellular approximation theorem

In classical algebraic topology, the cellular approximation theorem is a cornerstone result which states that any continuous function between two CW complexes is homotopic to a cellular map. This seems perfect for extending our constructions of pushouts to Ho(CW), but unfortunately this theorem appears to be out of reach in our constructive framework: the standard proof involves considerations of point-set topology and the use of the axiom of choice. However, what we can prove is a *synthetic* and *finitary* version of the theorem, which informally states that the cellular approximation theorem holds when the domain is finite dimensional. This will be the main result of this subsection.

Before providing the precise statement for our constructive cellular approximation theorem, let us start with a brief digression about finite subcomplexes and substructures – this will allow us to formulate a statement that is somewhat more flexible than the one suggested above. We say that a CW structure is *finite* (of dimension n) if the maps in its underlying sequence of types are equivalences starting from dimension n. Given any CW structure X_*, there is a canonical way to restrict it to a finite CW structure $X_*^{(n)}$ with the following definitions:

$$X_i^{(n)} := \begin{cases} X_i & \text{if } i < n \\ X_n & \text{otherwise} \end{cases} \qquad c_i^{(n)} := \begin{cases} c_i & \text{if } i \leq n \\ \bot & \text{otherwise.} \end{cases}$$

The structure $X_*^{(n)}$ trivially satisfies $X_\infty^{(n)} \simeq X_n$. We will use the same notation for cellular maps, writing $f_*^{(n)}$ for the restrictions of a cellular map f_* to the n-skeleton of the domain. For ease of notation, we also define $X_*^{(\infty)} := X_*$ (and similarly for $f_*^{(\infty)}$).

Definition 11. *Let X_* and Y_* be two CW structures, and let $f : X_\infty \to Y_\infty$ be an arbitrary map between their colimits. A **cellular n-approximation** of f is a cellular map $(f_*, h_*) : X_*^{(n)} \to Y_*$ along with a homotopy*

$$t : (x : X_n) \to f(\iota_\infty x) = \iota_\infty(f_n\, x).$$

Our first cellular approximation states that n-approximations always exist for n finite. To get there, we will need the help of two easy lemmas.

Lemma 12. *For any CW structure X_*, the inclusion map between two successive skeleta $\iota_i : X_i \to X_{i+1}$ is $(i-1)$-connected.*

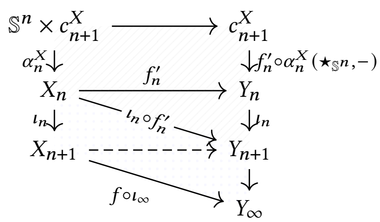
Proof. It is a general fact that given any span $B \xleftarrow{f} A \xrightarrow{g} C$, the map $\mathsf{inl} : B \to B \sqcup^A C$ is as connected as g [2, Proposition 2.3.10]. In our case, X_{i+1} is defined as the pushout of the span $X_i \leftarrow \mathbb{S}^i \times c_{i+1} \xrightarrow{\mathsf{snd}} c_{i+1}$, and thus it suffices to show that the projection $\mathsf{snd} : \mathbb{S}^i \times c_{i+1} \to c_{i+1}$ is $(i-1)$-connected. Indeed, its fibres are equivalent to $\mathbb{S}^i$, which is $(i-1)$-connected. ◻

Lemma 13. *For any CW structure X_*, the inclusion map into the colimit $\iota_\infty : X_i \to X_\infty$ is $(i-1)$-connected.*

Proof. It follows immediately from Lemma 12 that all of the maps $X_{i+k} \xrightarrow{\iota_{i+k}} X_{i+k+1}$ are at least $(i-1)$ connected. As a consequence, their transfinite composition $\iota_\infty : X_i \to X_\infty$ is also $(i-1)$-connected [18, Corollary 7.7]. ◻

Theorem 14 (First cellular approximation theorem). *Let X_* and Y_* be CW structures. For any map $f : X_\infty \to Y_\infty$ and $n : \mathbb{N}$, there merely exists a cellular n-approximation of f.*

Proof. The proof proceeds by induction on n. The base case, $n = -1$, is trivial. For the inductive step, assume that we have an n-approximation f'_* of f. We will use f'_* to *merely* construct an $(n+1)$-approximation $f_* : X_*^{(n+1)} \to Y_*$ (the fact that we are only aiming for mere existence allows us to use the elimination rule for propositional truncations a finite number of times). We define $f_i :\equiv f'_i$ for all $i \le n$. It remains to define f_{n+1} and its associated homotopies. Consider the following (not necessarily commutative) diagram:



If we can construct f_{n+1} as the dashed map above in a way that makes all triangles commute, we are done. By the elimination principle of pushouts, the dashed map exists if we can fill the shaded area. In other words, we need to construct an element of type $\|((x, y) : \mathbb{S}^n \times c_{n+1}) \to F(x, y) = F(\star, y)\|_{-1}$ for $F :\equiv \iota_n \circ f'_n \circ \alpha^X_n$. Using the projectivity of c_{n+1} and Lemma 4, this corresponds to constructing, for every $y : c_{n+1}$, a family of paths $(x : \mathbb{S}^n) \to \|F(x, y) = F(\star, y)\|_{n-1}$. By Lemma 13, the map $\iota_\infty : Y_{n+1} \to Y_\infty$ is n-connected and therefore its action on path spaces, $\mathsf{ap}_{\iota_\infty}$, is $(n-1)$-connected. Thus $\|F(x, y) = F(\star, y)\|_{n-1}$ is equivalent to $\|\iota_\infty(F(x, y)) = \iota_\infty(F(\star, y))\|_{n-1}$. Since the dotted area of the diagram commutes, it suffices to show that the outermost diagram commutes, which is a consequence of the identity $\iota_\infty \circ f'_n = f \circ \iota_\infty$ and the fact that X_* is a CW structure. The remaining homotopy involved in the definition of a cellular approximation holds by construction. □

Corollary 15. *For any span of CW complexes* $Y \xleftarrow{f} X \xrightarrow{g} Z$ *with* X *finite, the pushout* $Y \sqcup^X Z$ *is a CW complex.*

Unfortunately, our finitary approximation theorem is not quite strong enough to prove the existence of all pushouts in $\mathsf{Ho}(\mathsf{CW})$. One option to remedy this would be to assume the axiom of countable choice, which allows us to deduce the mere existence of an ∞-approximation from the mere existence of an n-approximation for every n. This would, however, limit the generality of our theorems (countable choice does not hold in arbitrary infinity toposes), so we will refrain from doing so.

Question 16. *Can we prove that every map between CW complexes merely has an* ∞*-approximation without using the axiom of countable choice?*

Since we do not know the answer to this question, we will have to work with finite cellular approximations for the rest of this paper. For this purpose, we introduce the notion of n-truncated complexes, which can be faithfully captured by finite cellular approximations:

Definition 17. *An* n*-truncated CW complex is an* n*-truncated type* A *for which there merely exists a CW structure* X_* *of dimension* $n+1$ *such that* $A \simeq \|X_{n+1}\|_n$.

We write $\mathsf{Ho}(\mathsf{CW}^{(n)})$ for the category of n-truncated CW complexes and ordinary maps. Note the mismatch between the dimension of the structure and the truncation level in Definition 17. This mismatch is here so that we may define a truncation functor trunc_n from $\mathsf{Ho}(\mathsf{CW})$ to $\mathsf{Ho}(\mathsf{CW}^{(n)})$: we can send the pair $(A, |X_*|)$ to $(\|A\|_n, |X_*^{(n+1)}|)$, and the isomorphism condition holds because $\|X_\infty\|_n \simeq \|X_{n+1}\|_n$. We also introduce a corresponding category $\mathsf{CWstr}^{(n)}$ whose objects are CW structures of dimension $n+1$, and whose morphisms are cellular maps of dimension n.

3.3 Cellular homotopies and the second approximation theorem

In essence, the first cellular approximation tells us that there *merely* exists an inverse to the colimit operation for finite cellular maps. This already lets us transfer some constructions from CWstr to $\mathsf{Ho}(\mathsf{CW})$, but we would ideally like to get rid of that propositional truncation and define a proper approximation functor from $\mathsf{Ho}(\mathsf{CW})$ to CWstr – or at least from $\mathsf{Ho}(\mathsf{CW}^{(n)})$ to $\mathsf{CWstr}^{(n)}$, to avoid choice issues. Unfortunately, this turns out to be problematic, as the cellular approximation theorem is inconsistent without the propositional truncation.

Theorem 18. *The set-truncated version of Theorem 14 is false.*

Proof. Both $\mathbb{1}$ and $\mathbb{S}^1$ can be presented by finite CW structures (which we will denote by $\mathbb{1}_*$ and ${\mathbb{S}^1}_*$), with only a 0-cell for the former and a 0-cell plus a 1-cell for the latter. Given any $x : \mathbb{S}^1$, define $\widehat{x} : \mathbb{1} \to \mathbb{S}^1$ to be the corresponding function. If we have a cellular approximation of $\widehat{x}$, then we have a factorisation

$$\mathbb{1} \xrightarrow{\sim} \mathbb{1}_0 \xrightarrow{\widehat{x}_0} {\mathbb{S}^1}_0 \to \mathbb{S}^1$$

which in turn implies that x is equal to the basepoint $\star_{\mathbb{S}^1}$. Therefore, the set-truncated statement of Theorem 14 provides us with a proof of $(x : \mathbb{S}^1) \to \|x = \star_{\mathbb{S}^1}\|_0$. By Lemma 4, this entails that $\|(x : \mathbb{S}^1) \to x = \star_{\mathbb{S}^1}\|_{-1}$ which by truncation elimination implies that $\mathbb{S}^1$ is contractible. But this is provably false in HoTT [11]. □

This problem stems from a mismatch between the notion of equality for morphisms in $\mathsf{Ho}(\mathsf{CW})$ and the notion of equality for morphisms in CWstr. Since any homotopy gives rise to an equality between maps, the morphisms in $\mathsf{Ho}(\mathsf{CW})$ should be understood as maps *up to homotopy*, while the equality between morphisms in CWstr is much closer in spirit to a strict equality. Therefore, if we want to frame our approximation theorem as a functor, we need to quotient the morphisms of the target category by an adequate notion of homotopy.

Definition 19. *A* ***cellular homotopy*** *between cellular maps* $f_*, g_* : X_* \to Y_*$ *is a family*

$$p_i : (x : X_i) \to \iota_i(f_i(x)) =_{Y_{i+1}} \iota_i(g_i(x))$$

with fillers $q_i\,x$*, for each* $i \ge 0$ *and* $x : X_i$*, of the following square*

$$\begin{array}{ccc} \iota_{i+1}(\iota_i\,(f_i\,x)) & \xrightarrow{\mathsf{ap}_{\iota_{i+1}}(p_i\,x)} & \iota_{i+1}(\iota_i\,(g_i\,x)) \\ \downarrow & q_i\,x & \downarrow \\ \iota_{i+1}(f_{i+1}(\iota_i\,x)) & \xrightarrow[p_{i+1}(\iota_i\,x)]{} & \iota_{i+1}(g_{i+1}(\iota_i\,x)) \end{array}$$

We use the notation $(p_*, q_*) : f_* \sim g_*$ or simply $p_* : f_* \sim g_*$ when the q_i's are clear from context.

One can easily prove that composition of cellular maps is invariant with respect to cellular homotopy. This lets us define the category $\mathsf{Ho}(\mathsf{CWstr})$, whose objects are CW structures and whose morphisms are cellular maps up to cellular homotopy. Furthermore, the existence of a cellular homotopy between f_* and g_* implies that their colimits are homotopic, or in other words, that $f_\infty = g_\infty$. This means that the colimit functor factors through $\mathsf{Ho}(\mathsf{CWstr})$.

That new colimit functor *almost* induces an equivalence between the categories $\mathsf{Ho}(\mathsf{CW}^{(n)})$ and $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$. In order to prove this, we will need to extend our approximation theorem to cellular homotopies. Because the caveats regarding the axiom of countable choice still apply, we start by introducing a notion of finite approximation for cellular homotopies.

Definition 20. *Let $f_*, g_* : X_* \to Y_*$ be two cellular maps, and let $p : (x : X_\infty) \to f_\infty(x) = g_\infty(x)$ be a homotopy between their colimits. A **cellular n-approximation** of p is a cellular homotopy $p_* : f_*^{(n)} \sim g_*^{(n)}$ equipped with a filler of the following square for each $x : X_n$.*

$$\begin{array}{ccc} f_\infty(\iota_\infty\, x) & \xrightarrow{\;p(\iota_\infty\, x)\;} & g_\infty(\iota_\infty\, x) \\ \downarrow & & \downarrow \\ \iota_\infty(\iota_n(f_n(x))) & \xrightarrow[\mathsf{ap}_{\iota_\infty}(p_n(x))]{} & \iota_\infty(\iota_n(g_n(x))) \end{array}$$

We are now ready to state the second cellular approximation theorem. Its proof follows the same strategy as Theorem 14, so we omit it but remark that it has been mechanised.

Theorem 21 (Second cellular approximation theorem). *Let $f_*, g_* : X_* \to Y_\infty$ be cellular maps and $p : f_\infty \sim g_\infty$. For any $n : \mathbb{N}$, there merely exists an n-approximation of p.*

Theorem 21 implies that taking the colimit of a cellular map between two CW structures in $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$ is an injective operation. On the other hand, Theorem 14 implies that it is a surjective operation. Therefore, the colimit induces a fully faithful functor from $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$ to $\mathsf{Ho}(\mathsf{CW}^{(n)})$. Since this functor is essentially surjective in the sense of [19, Chapter 9], we get the following result as a corollary.

Corollary 22. *The colimit functor induces a weak equivalence between $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$ and $\mathsf{Ho}(\mathsf{CW}^{(n)})$. Equivalently, $\mathsf{Ho}(\mathsf{CW}^{(n)})$ is the Rezk completion of $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$.*

$$\begin{array}{ccccc} \mathsf{Ho}(\mathsf{CW}) & \xleftarrow{\;\mathsf{colim}\;} & \mathsf{Ho}(\mathsf{CWstr}) & \longleftarrow & \mathsf{CWstr} \\ \downarrow{\scriptstyle \mathsf{trunc}_n} & & \downarrow{\scriptstyle \mathsf{trunc}_n} & & \downarrow{\scriptstyle \mathsf{trunc}_n} \\ \mathsf{Ho}(\mathsf{CW}^{(n)}) & \xleftarrow[\mathsf{colim}]{\sim} & \mathsf{Ho}(\mathsf{CWstr}^{(n)}) & \longleftarrow & \mathsf{CWstr}^{(n)} \end{array}$$

Figure 1: The categories at play

The relations between the various categories defined so far are summarised in Figure 1. This diagram gives us a systematic way of lifting a functor F defined over CWstr to a functor defined over $\mathsf{Ho}(\mathsf{CW})$: first, if the functor F happens to use only a finite number of dimensions, it can be factored as $\overline{F} \circ \mathsf{trunc}_n$ for some functor $\overline{F}$ defined over $\mathsf{CWstr}^{(n)}$. Then, if we manage to prove that $\overline{F}$ is invariant under cellular homotopy, we can extend it to a functor $\widetilde{F}$ defined over $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$. Finally, if the target is a univalent category, $\widetilde{F}$ can be extended to a functor defined over the Rezk completion of $\mathsf{Ho}(\mathsf{CWstr}^{(n)})$, which is $\mathsf{Ho}(\mathsf{CW}^{(n)})$. By composing the result with the truncation functor, we get a lift of F to $\mathsf{Ho}(\mathsf{CW})$.

4 CELLULAR HOMOLOGY

In algebraic topology, the homology groups of a space is a family of topological invariants which are somewhat similar to homotopy groups in that they intuitively measure the number of n-dimensional holes, but are much simpler to compute. There is a plethora of *homology theories* (roughly, different definitions for these homology groups) but among them, one is especially relevant to our interests: the theory of *cellular homology* is defined in terms of CW structures, and is particularly well suited for computation. This makes it all the more interesting to translate to HoTT, and especially to one of its computational implementations such as `Cubical Agda`: through the Curry–Howard correspondence, a mechanisation of cellular homology would automatically provide formally verified computations of homology groups, facilitating the idea of 'proof by computation' – a central idea in computer formalisation of synthetic homotopy theory [2, 3, 10, 12].

In this section, we start by defining (reduced) homology as a family of functors $\widetilde{H}_i^{\mathsf{str}} : \mathsf{CWstr} \to \mathsf{AbGrp}$. Then, we lift this definition to a family of functors $\widetilde{H}_i^{\mathsf{cw}}$ defined over $\mathsf{Ho}(\mathsf{CW})$, using our freshly proved cellular approximation theorem. This provides the first definition of cellular homology in HoTT.

4.1 A crash course in homological algebra

The first step in the definition of homology groups is to approximate CW structures by *cellular chain complexes.* Before doing so, however, we need some preliminary background on chain complexes, as well as a definition of the homology groups of a chain complex.

Definition 23. *A **chain complex** is a sequence of abelian groups (called i-chains)*

$$\ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} \ldots$$

where the maps ∂_i (called boundary maps) are group homomorphisms satisfying the equation $\partial_i \circ \partial_{i+1} = 0$.

Definition 24. *A **chain map** $\phi_* : C_* \to D_*$ is a collection of group homomorphisms $\phi_i : C_i \to D_i$ compatible with boundary maps in the sense that $\phi_i \circ \partial^C_{i+1} = \partial^D_{i+1} \circ \phi_{i+1}$.*

There are natural definitions of chain map composition (levelwise composition) and of the identity chain map (the levelwise identity). This lets us define the category Ch whose objects are chain complexes and whose morphisms are chain maps. We also have a natural notion of chain homotopy.

Definition 25. *A **chain homotopy** η_* between two chain maps $\phi_*, \psi_* : C_* \to D_*$ is a sequence of homomorphisms $\eta_i : C_i \to D_{i+1}$ such that $\phi_i - \psi_i = \partial^D_{i+1} \circ \eta_i + \eta_{i-1} \circ \partial^C_i$.*

Chain homotopies are compatible with composition, which lets us define the homotopy category of chain complexes $\mathsf{Ho}(\mathsf{Ch})$ whose morphisms are chain maps up to chain homotopy. We finally arrive at the definition of homology groups, which is the natural analogue of homotopy groups in the category of chain complexes.

Definition 26 (Homology groups). *We define the nth homology group of a chain complex* (C_*, ∂_*) *by* $H_n(C_*) := \ker \partial_n / \operatorname{im} \partial_{n+1}$.

We remark that the quotient in the definition above is well-defined since the boundary equation $\partial_i \circ \partial_{i+1} = 0$ ensures that $\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i$. Furthermore, any chain map $\phi_* : C_* \to D_*$ induces a homomorphism $H_n(\phi_*) : H_n(C_*) \to H_n(D_*)$, and it does so in a functorial way. Thus, the definition of the nth homology group can be presented as a functor from Ch to the category of abelian groups AbGrp. Lastly, a standard argument shows that the existence of a chain homotopy between two chain maps ϕ_* and ψ_* implies that $H_n(\phi_*) \cong H_n(\psi_*)$. Therefore, the definition of H_n factors through the category $\mathrm{Ho}(\mathrm{Ch})$. This concludes our definition of the homology groups of a chain complex.

4.2 Sphere bouquets and reduced cellular homology

We are now in a position to define the cellular chain complex associated to a CW structure X_*. The definition for the abelian groups of n-chains is rather straightforward:

- when $n \geq 0$, we set $C_n := \mathbb{Z}[c_n^X]$, i.e. C_n is the free abelian group with a generator for each n-cell in X_*,
- when $n = -1$, we set $C_{-1} := \mathbb{Z}$ (in technical terms, this means that we are defining the *augmented* chain complex of X_*, but we will not go into detail here),
- when $n < -1$, we define C_n to be the trivial group.

The definition of the boundary maps is slightly more involved. In positive degrees, our goal is to construct a homomorphism of free abelian groups $\partial_{i+1} : \mathrm{Hom}(\mathbb{Z}[c_{i+1}^X], \mathbb{Z}[c_i^X])$. To do so, we will exploit the fact that free abelian groups are closely related to wedge sums of spheres, which we call *sphere bouquets*. This approach is essentially a reinterpretation of the definition used by May [14] and Buchholtz and Favonia [4].

Definition 27. *Given* A : pSet *and an integer* $n \geq 0$*, we define the* ***sphere bouquet*** *over* A *and dimension* n *to be the type* $\bigvee_A \mathbb{S}^n$*, i.e. the wedge sum of 'A copies' of* n*-spheres.*

Before clarifying the relation between sphere bouquets and free abelian groups, we first need to recall some well-known facts about the *degree* of an endofunction of $\mathbb{S}^n$. For any $n > 0$, there is an isomorphism deg from $\pi_n(\mathbb{S}^n)$ to $\mathbb{Z}$. In fact, this definition extends to any (not necessarily pointed) map $f : \mathbb{S}^n \to \mathbb{S}^n$. This is done by noting that the 'forgetful map' $\|\mathrm{fst}\|_0 : \pi_n(\mathbb{S}^n) \to \|\mathbb{S}^n \to \mathbb{S}^n\|_0$ is an equivalence. This allows us to define a degree map by the composition

$$(\mathbb{S}^n \to \mathbb{S}^n) \xrightarrow{|-|} \left\|\mathbb{S}^n \to \mathbb{S}^n\right\|_0 \xrightarrow{\|\mathrm{fst}\|_0^{-1}} \left\|\mathbb{S}^n \to_\star \mathbb{S}^n\right\|_0 \xrightarrow{\deg} \mathbb{Z}$$

We allow some overloading of notation by also using deg to denote the above composition. In addition to inducing an isomorphism of groups, deg has a few useful properties.

Proposition 28. *The degree map commutes with suspensions, i.e. any function* $f : \mathbb{S}^n \to \mathbb{S}^n$ *is of the same degree as its suspension* $\Sigma f : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$*. Additionally,* deg *takes function composition to integer multiplication, i.e.* $\deg(f \circ g) = \deg f \cdot \deg g$.

The degree map has been well studied in HoTT already, and the proof of Proposition 28 can be found in *e.g.* [4, 5]. Thus, we can use this map to recover a homomorphism of type $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ from a function of type $\mathbb{S}^n \to \mathbb{S}^n$. Now, we would like to generalise this scenario a little bit and find a way to extract homomorphisms of type $\mathrm{Hom}(\mathbb{Z}[A], \mathbb{Z}[B])$ from maps between sphere bouquets $\bigvee_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n$. If we had decidable equality on B, we could use the degree map for this purpose, but we cannot assume this in a constructive setting. [4] Since we cannot reuse the degree map, our only option is to go back to square one and compute $\pi_n(\bigvee_B \mathbb{S}^n)$ or, rather, $\pi_n^{\mathrm{ab}}(\bigvee_B \mathbb{S}^n)$ – we need to abelianise to avoid technicalities when $n = 1$.

Proposition 29. *Given* B : pSet *and an integer* $n \geq 1$*, we have*

$$\pi_n^{\mathrm{ab}}(\textstyle\bigvee_B \mathbb{S}^n) \simeq \mathbb{Z}[B]$$

Proof sketch. The case $n = 1$ follows from the Seifert–Van Kampen theorem [19, Section 8.7], and the cases $n > 2$ follow from the Freudenthal suspension theorem [19, Theorem 8.6.4]. It only remains to handle the case $n = 2$: first, we use an encode-decode proof to show that $\|\bigvee_B \mathbb{S}^2\|_2 \simeq K(\mathbb{Z}[B], 2)$. From there, it follows that $\pi_2(\bigvee_B \mathbb{S}^2) \simeq \pi_2(K(\mathbb{Z}[B], 2)) \simeq \mathbb{Z}[B]$. Finally, a close look at the construction reveals that this equivalence is a group isomorphism. ◻

From this computation, we deduce that (assuming $n \geq 2$ to avoid abelianisation)

$$\begin{aligned} \|\textstyle\bigvee_A \mathbb{S}^n \to_\star \bigvee_B \mathbb{S}^n\|_0 &\simeq \textstyle\bigoplus_A \|\mathbb{S}^n \to_\star \bigvee_B \mathbb{S}^n\|_0 \\ &\simeq \textstyle\bigoplus_A \mathbb{Z}[B] \\ &\simeq \mathrm{Hom}(\mathbb{Z}[A], \mathbb{Z}[B]) \end{aligned}$$

where the last equivalence is defined using the universal property of the free abelian group. Just like before, this induces a map that we call the *bouquet degree*

$$\mathrm{bdeg} : \|\textstyle\bigvee_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n\|_0 \to \mathrm{Hom}(\mathbb{Z}[A], \mathbb{Z}[B]).$$

The bouquet degree map immediately inherits properties corresponding to those listed in Proposition 28:

Proposition 30. *Let* A, B *and* C *be projective types,* $n \geq 0$*. The following facts hold.*

(1) *The bouquet degree function induces a group homomorphism* $\left\|\bigvee_A \mathbb{S}^{n+1} \to \bigvee_B \mathbb{S}^{n+1}\right\|_0 \to \mathrm{Hom}(\mathbb{Z}[A], \mathbb{Z}[B])$*, where the group structure on the left-hand side is the natural extension of the group structure on* $\pi_{n+1}(\mathbb{S}^{n+1})$.

[4]This is the place where using finite sets in the formalisation becomes useful. In this case, we can give bdeg a rather explicit construction (and, hopefully, with better computational properties) as follows:

$$\begin{array}{ccc} (\bigvee_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n) & & \Pi_A \Pi_B (\mathbb{S}^n \to \mathbb{S}^n) \\ \downarrow & & \downarrow (\deg^*)^* \\ (\Pi_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n) & & \Pi_A \Pi_B \mathbb{Z} \\ \iota_\vee{}^* \downarrow & & \downarrow \wr \\ (\Pi_A \mathbb{S}^n \to \Pi_B \mathbb{S}^n) & \nearrow & \mathrm{Hom}(\mathbb{Z}[A], \mathbb{Z}[B]) \end{array}$$

This does not work when B is an arbitrary projective set since, in this case, we do not have an isomorphism between $(B \to \mathbb{Z})$ and $\mathbb{Z}[B]$.

(2) *The bouquet degree function commutes with suspension, i.e. any* $f : \bigvee_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n$ *is of the same degree as its suspension* $\Sigma f : \Sigma(\bigvee_A \mathbb{S}^n) \to \Sigma(\bigvee_B \mathbb{S}^n)$*, where the bouquet degree of the latter function is well-defined since* $\Sigma(\bigvee_X \mathbb{S}^n) \simeq \bigvee_X \mathbb{S}^{n+1}$ *for any* X.

(3) *The bouquet degree function respects composition, i.e. for* $f : \bigvee_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n$ *and* $g : \bigvee_B \mathbb{S}^n \to \bigvee_C \mathbb{S}^n$ *we have* $\mathsf{bdeg}(g \circ f) = \mathsf{bdeg}\, g \circ \mathsf{bdeg}\, f$.

We can now return to the construction of the boundary maps: we would like to define, for any CW structure X_*, a homomorphism $\partial_{i+1} : \mathbb{Z}[c^X_{i+1}] \to \mathbb{Z}[c^X_i]$. By applying our bouquet degree function, it suffices to construct a function $d_i : \bigvee_{c^X_{i+1}} \mathbb{S}^{i+1} \to \bigvee_{c^X_i} \mathbb{S}^{i+1}$. We recall from [4] that there is an equivalence $e : X_i/X_{i-1} \simeq \bigvee_{c^X_i} \mathbb{S}^i$. When $i > 0$, we obtain it by considering the following diagram.

$$\begin{array}{ccccc} \mathbb{S}^{i-1} \times c^X_i & \xrightarrow{\alpha^X_{i-1}} & X_{i-1} & \longrightarrow & \mathbb{1} \\ \downarrow & & \downarrow & & \downarrow \\ c^X_i & \longrightarrow & X_i & \longrightarrow & \Sigma(\bigvee_{c^X_i} \mathbb{S}^{i-1}) \end{array}$$

Since the outermost square is a pushout, we know, by pushout pasting, that so is the right square. The construction of e is completed by observing that suspension commutes with wedge sums. When $i = 0$, the equivalence is obtained by noting that $X_0/X_{-1} \simeq X_0 + \mathbb{1}$ which allows us to identify the appended point with the basepoint in $\bigvee_{c^X_0} \mathbb{S}^0$. We may now construct the desired map d_{i+1} by considering the composition

$$\bigvee_{c^X_{i+1}} \mathbb{S}^{i+1} \xrightarrow{\sim} X_{i+1}/X_i \xrightarrow{\mathsf{pinch}} \Sigma X_i \xrightarrow{\Sigma\mathsf{cfcod}} \Sigma(X_i/X_{i-1}) \xrightarrow{\sim} \bigvee_{c^X_i} \mathbb{S}^{i+1}$$

where pinch is the *pinch* map, i.e. the pointed map identifying inr with south and push with merid. Finally, we set $\partial_{i+1} = \mathsf{bdeg}\, d_{i+1}$. Note that this whole construction is only valid for $i > -1$. To complete the definition, we define $\partial_0 : \mathbb{Z}[c^X_0] \to \mathbb{Z}$ by sending every generator of $\mathbb{Z}[c^X_0]$ to 1, and lastly we let the maps in negative dimension be trivial.

Proposition 31. *The boundary maps satisfy* $\partial_i \circ \partial_{i+1} = 0$.

Proof. First, assume that $i > 0$. We compute:

$$\begin{aligned} \partial_i \circ \partial_{i+1} = \mathsf{bdeg}\, d_i \circ \mathsf{bdeg}\, d_{i+1} &= \mathsf{bdeg}\,(\Sigma d_i) \circ \mathsf{bdeg}\,(d_{i+1}) \\ &= \mathsf{bdeg}\,(\Sigma d_i \circ d_{i+1}) \end{aligned}$$

We are done if we can show that $\Sigma d_i \circ d_{i+1} = 0$. This composition of maps is defined as follows.

$$\begin{array}{l} \bigvee_{c^X_{i+1}} \mathbb{S}^{i+1} \longrightarrow X_{i+1}/X_i \longrightarrow \Sigma X_i \dashrightarrow \Sigma(X_i/X_{i-1}) \dashrightarrow \bigvee_{c^X_i} \mathbb{S}^{i+1} \\ \dashrightarrow \Sigma \bigvee_{c^X_i} \mathbb{S}^i \dashrightarrow \Sigma(X_i/X_{i-1}) \dashrightarrow \Sigma^2 X_{i-1} \to \Sigma^2(X_{i-1}/X_{i-2}) \to \Sigma \bigvee_{c^X_{i-1}} \mathbb{S}^i \end{array}$$

It is enough to show that the dashed composition $\Sigma X_i \to \Sigma^2 X_{i-1}$ is trivial. By tracing the construction of the maps involved, it is easy to see that the map is given by

$$\Sigma X_i \xrightarrow{\Sigma\mathsf{cfcod}} \Sigma X_i/X_{i-1} \xrightarrow{\Sigma\mathsf{pinch}} \Sigma^2 X_{i-1}$$

which is equal to functorial action of Σ on $\mathsf{pinch} \circ \mathsf{cfcod} : X_i \to \Sigma X_{i-1}$. This is constant by definition. The case $i = 0$ follows by an explicit computation of the maps ∂_1 and ∂_0. □

At this point, we have a proper definition for the cellular chain complex of a CW structure. It remains to show that this construction lifts to a functor from CWstr to Ch.

Let $f_* : X_* \to Y_*$ be a cellular map. Because f_* is cellular, it determines a map $X_i/X_{i-1} \to Y_i/Y_{i-1}$. With a bit of help from the equivalence e that we defined earlier, we can define a map of sphere bouquets $\widetilde{f_i}$ as follows:

$$\bigvee_{c^X_i} \mathbb{S}^i \xrightarrow{e^{-1}} X_i/X_{i-1} \xrightarrow{f_i/f_{i-1}} Y_i/Y_{i-1} \xrightarrow{e} \bigvee_{c^Y_i} \mathbb{S}^i.$$

We may thus define the functorial action of f on i-chains

$$\overline{f}_i : \mathsf{Hom}(\mathbb{Z}[c^X_i], \mathbb{Z}[c^Y_i])$$

by setting $\overline{f}_i = \mathsf{bdeg}\, \widetilde{f_i}$. Let us verify that it is a chain map, i.e. that $\partial_{i+1} \circ \overline{f}_{i+1} = \overline{f}_i \circ \partial_{i+1}$. Using the fact that bdeg respects suspension and composition, this is equivalent to $\mathsf{bdeg}\,(d_{i+1} \circ \widetilde{f_{i+1}}) = \mathsf{bdeg}\,(\Sigma\widetilde{f_i} \circ d_{i+1})$. Let us simply show that $d_{i+1} \circ \widetilde{f_{i+1}} = \Sigma\widetilde{f_i} \circ d_{i+1}$. That is, we will show that the outer square commutes in the diagram below:

$$\begin{array}{ccccccccc} \bigvee_{c^X_{i+1}} \mathbb{S}^{i+1} & \longrightarrow & X_{i+1}/X_i & \longrightarrow & \Sigma X_i & \longrightarrow & \Sigma(X_i/X_{i-1}) & \longrightarrow & \bigvee_{c^X_i} \mathbb{S}^{i+1} \\ \downarrow{\scriptstyle \widetilde{f_{i+1}}} & & \downarrow{\scriptstyle f_{i+1}/f_i} & & \downarrow{\scriptstyle \Sigma f_i} & & \downarrow{\scriptstyle \Sigma f_i/f_{i-1}} & & \downarrow{\scriptstyle \Sigma\widetilde{f_i}} \\ \bigvee_{c^Y_{i+1}} \mathbb{S}^{i+1} & \longrightarrow & Y_{i+1}/Y_i & \longrightarrow & \Sigma Y_i & \longrightarrow & \Sigma(Y_i/Y_{i-1}) & \longrightarrow & \bigvee_{c^Y_i} \mathbb{S}^{i+1} \end{array}$$

This is immediate: the leftmost and rightmost squares commute by construction of our functorial action, and the middle squares commute by definition.

Thus, we have shown that any cellular map $f_* : X_* \to Y_*$ gives rise to a chain map between the cellular chain complexes of X_* and Y_*. Due to space constraints, we omit the proofs that this operation satisfies the two functor axioms, but we note that they are very direct. This results in a functor $\mathsf{cellChain} : \mathsf{CWstr} \to \mathsf{Ch}$. If we compose this functor with the nth homology functor H_n : Ch → AbGrp, we obtain a functorial definition of reduced cellular homology for CW structures. We denote the resulting functor by $\widetilde{H}^{\mathsf{str}}_n : \mathsf{CWstr} \to \mathsf{AbGrp}$.

4.3 The homology of a CW complex

Our end goal is to extend our cellular homology functor to the category of CW complexes. To do so, we follow the strategy laid out in Figure 1: first, we will need a lemma to show that cellular homology is homotopy invariant.

Proposition 32. *Let* f_* *and* g_* *be two parallel cellular maps. Every cellular homotopy between* f_* *and* g_**, induces a chain homotopy between* $\mathsf{cellChain}(f_*)$ *and* $\mathsf{cellChain}(g_*)$.

The proof is standard but somewhat technical. Due to space constraints, we omit it and refer to the computer formalisation. Proposition 32 implies that cellChain descends to a functor from Ho(CWstr) to Ho(Ch). As we already saw, the chain homology functor H_n factors through Ho(Ch), meaning that we can compose it with cellChain to express cellular homology as a functor $\widetilde{H}^{\mathsf{str}}_n$: Ho(CWstr) → AbGrp. Therefore, we have established that cellular homology is homotopy invariant.

In fact, a quick glance at the definition of cellular homology makes it clear that $\widetilde{H}^{\mathsf{str}}_n(X_*)$ only depends on the $(n+1)$-skeleton of X_*, so $\widetilde{H}^{\mathsf{str}}_n$ can actually be defined as a functor from $\mathsf{Ho}(\mathsf{CWstr}^{(n+1)})$ to AbGrp. Since abelian groups form a univalent category, $\widetilde{H}^{\mathsf{str}}_n$ can even be extended to the Rezk completion of $\mathsf{Ho}(\mathsf{CWstr}^{(n+1)})$, which is $\mathsf{Ho}(\mathsf{CW}^{(n+1)})$. Composing the resulting functor with the truncation functor from $\mathsf{Ho}(\mathsf{CW})$ to $\mathsf{Ho}(\mathsf{CW}^{(n+1)})$ yields the desired definition of the cellular homology functor $\widetilde{H}^{\mathsf{cw}}_n : \mathsf{Ho}(\mathsf{CW}) \to \mathsf{AbGrp}$.

Remark 33. If we want to be thorough, we should prove that our cellular homology functors satisfy the *Eilenberg–Steenrod axioms*. These four simple axioms completely characterise the homology groups of CW complexes, and in order to deserve the name of *homology theories*, $\widetilde{H}^{\mathsf{str}}_n$ and $\widetilde{H}^{\mathsf{cw}}_n$ should satisfy them.

Given that our construction of cellular homology follows the classical recipe (as presented by May [14], for instance), we can replicate a textbook proof and the arguments carry over without too much difficulty. Some caveats apply, of course: the wedge axiom is restricted to *projective* index sets, and in the case of $\widetilde{H}^{\mathsf{cw}}_n$, we can only show a finitary version of exactness as our construction of pushouts only supports finite-dimensional domains.

5 THE HUREWICZ THEOREMS

As previously mentioned, homology groups are quite similar in spirit to homotopy groups, so one might hope that the two notions are connected in some way. The answer lies in the Hurewicz theorem, which states that if a space is n-connected, then its homology groups coincide with its homotopy groups up to dimension $n+1$ (up to abelianisation in the case $n=0$). This theorem, which has already been proved in HoTT by Christensen and Scoccola [7], is not our main contribution here. Rather, we prove a fundamental result concerning the equivalence of two alternative definitions of n-connected CW complexes – this result happens to give Hurewicz theorem for CW complexes as a corollary, and thus we also provide a novel proof of this theorem.

5.1 Approximating n-connected spaces

The classical proof of the Hurewicz theorem for cellular homology takes an arbitrary n-connected CW complex, and replaces its CW structure with an alternative one with no non-trivial cells in dimension $< n+1$. This is done by defining the new set of $(n+1)$ cells to be the generators of the $(n+1)$-fst homotopy group of the space, from which the Hurewicz theorem will follow. Unfortunately, this approach will not work in our framework since the homotopy groups of CW complexes are not necessarily finitely generated – this applies to, for instance, $\pi_2(\mathbb{S}^1 \vee \mathbb{S}^2)$. Yet, perhaps surprisingly, we are able to give a constructive proof of the Hurewicz theorem by using a different construction for the alternative structure of n-connected CW complexes.

Definition 34. *We say that a CW structure X_* is* ***Hurewicz n-connected*** *if $|c^X_0| = 1$ and $|c^X_i| = 0$ for $0 < i \leq n$. We use the same terminology for CW complexes which merely have a Hurewicz n-connected CW structure.*

We remark that being Hurewicz n-connected is a property (i.e. a proposition). The following lemma gives a few elementary consequences of Hurewicz n-connectedness.

Lemma 35. *Let X_* be a CW structure. If X_* is Hurewicz n-connected, then*

(1) $X_i \simeq \mathbb{1}$ for $0 \leq i \leq n$,
(2) $X_{n+1} \simeq \bigvee_{c^X_{n+1}} \mathbb{S}^{n+1}$,
(3) X_i is n-connected for $i \in \mathbb{N} \cup \{\infty\}$.

Item 3 tells us that Hurewicz n-connectedness implies the usual notion of n-connectedness. The other direction is much less obvious – especially constructively. Nonetheless, we can, in fact, prove it. In order to provide some some intuition for why it is true, let us consider the case when $n=0$ and the CW complex has finite cells.

Proposition 36. *For any 0-connected structure X_* with finite cells, there is a Hurewicz 0-connected CW structure X'_* s.t. $X_i = X'_i$ for $i \geq 1$.*

Proof. We proceed by induction on $|c^X_0|$, i.e. the size of X_0 (indeed, we have $X_0 \simeq c^X_0$). If $|c^X_0| = 0$, this contradicts the 0-connectedness of X_*. If $|c^X_0| = 1$, then X_* is already of the right form and there is nothing to prove. Consider now the case $|c^X_0| > 1$. We will be done if we can show that X_1 may be obtained as the pushout of $c'_0 \xleftarrow{\alpha'} \mathbb{S}^0 \times c'_1 \xrightarrow{\mathsf{snd}} c'_1$ for some α' and some finite sets c'_0 and c'_1 satisfying $|c'_0| < |c^X_0|$. Let us carry out the construction. Some of the arguments may look non-constructive but we emphasise that they are justified as they concern finite sets.

First, note that there must be some $a_0 : c_1$ such that $\alpha_0(\mathsf{north}, a_0) \neq \alpha_0(\mathsf{south}, a_0)$. Indeed, if this were not the case, we would have that $\|X_1\|_0 \simeq X_0$. By combining this equation with $\|X_\infty\|_0 \simeq \|X_1\|_0$, we would obtain that $\|X_\infty\|_0$ is isomorphic to X_0, which is not contractible since $|c^X_0| > 1$. Now, by permuting the elements of c_1 and c_0, we may assume that the last element $a_0 : c_1$ satisfies $\alpha_0(\mathsf{north}, a_0) = |c^X_0| - 1$ and $\alpha_0(\mathsf{south}, a_0) = |c^X_0| - 2$. We define a new attaching map $\alpha'_0 : \mathbb{S}^0 \times (c^X_1 - 1) \to (c^X_0 - 1)$ by

$$\alpha'_0(x,y) = \begin{cases} \alpha_0(x,y) \text{ if } \alpha_0(x,y) < |c_0| - 1 \\ |c^X_0| - 2 \text{ otherwise} \end{cases}$$

The 1-skeleton X'_1 obtained by pushing out along α'_0 is easily identified with X_1, and thus we are done as we have decreased the cardinality of the codomain of the attaching map by 1. □

Before turning to the general case, let us define a useful alteration of the notion of a CW structure. In what follows, we abuse notation for the sake of convenience and interpret $\bigvee_A \mathbb{S}^{-1}$ as the empty type rather than than the unit type.

Definition 37. *A* ***good*** *CW structure is a pointed CW structure X_* whose attaching maps $\alpha_i : c^X_{i+1} \times \mathbb{S}^i \to X_i$ lift to maps defined over sphere bouquets, i.e. for all i, there is a matching $\alpha'_i : \bigvee_{c^X_{i+1}} \mathbb{S}^i \to_\star X_i$ such that $X_{i+1} \simeq C_{\alpha'_i}$.*

Lemma 38. *Let X_* be a good CW structure. X_* is Hurewicz n-connected iff $X_{n+1} \simeq \bigvee_B \mathbb{S}^{n+1}$ and $X_{n+2} \simeq C_f$ where A and B are finite types and $f : \bigvee_A \mathbb{S}^{n+1} \to \bigvee_B \mathbb{S}^{n+1}$.*

This lemma follows immediately from the definition of good structures and Lemma 35. We remark that good CW structures always are Hurewicz 0-connected. The converse also holds for connectedness reasons.

Proposition 39. *Any Hurewicz 0-connected CW structure is merely good.*

We are now ready to prove the main technical theorem of which states that the synthetic standard notion of connectedness coincides, for CW complexes, with the more analytic notion of Hurewicz connectedness.

Theorem 40. *Let X_* be an n-connected CW structure. There merely exists a Hurewicz n-connected CW structure X'_* such that $X_i = X'_i$ for $i > n$.*

Proof. We proceed by induction on n. The base case follows by drawing the same diagrams as in the inductive step, and is close to identical, so we omit it. For the inductive step, let X_* be n-connected. In particular, X_* is $(n-1)$-connected, so by induction hypothesis we may assume that it is Hurewicz $(n-1)$-connected. Since $n > 0$, this structure is also Hurewicz 0-connected and we may assume that it is good (up to some fixed finite dimension $k \gg n$) by Proposition 39. Using Lemma 38, we know that $X_n \simeq \bigvee_A \mathbb{S}^n$ and $X_{n+1} \simeq C_f$ for $f : \bigvee_B \mathbb{S}^n \to \bigvee_A \mathbb{S}^n$ where A, B : pSet. Using Lemma 38 again, we are done if we can construct sets A', B' and $f' : \bigvee_{B'} \mathbb{S}^{n+1} \to \bigvee_{A'} \mathbb{S}^{n+1}$ s.t. $X_{n+2} \simeq C_{f'}$ (note that we are then implicitly setting $X'_{n+1} := \bigvee_{A'} \mathbb{S}^{n+1}$). Consider the following diagram where $C = c^X_{n+2}$.

$$\begin{array}{ccccc} & & \bigvee_C \mathbb{S}^{n+1} & \longrightarrow & \mathbb{1} \\ & & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow \\ \bigvee_A \mathbb{S}^n & \xrightarrow{\mathsf{cfcod}} & C_f & \longrightarrow & X_{n+2} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{1} & \longrightarrow & \bigvee_B \mathbb{S}^{n+1} & \longrightarrow & X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1} \end{array}$$

The top square is a pushout square because $X^{(k)}_*$ is a good CW structure (we have identified X_{n+1} with C_f). The fact that the bottom-left square is a pushout follows by the first part of Lemma 2. The bottom right-square is less evident. Consider the composite map $\bigvee_A \mathbb{S}^n \to X_{n+2}$ on the second row. Using the fact that X_∞ (and hence also X_{n+2}) is n-connected, it is an easy consequence of Lemma 4 that this map is merely constant. As we are proving a proposition, we may ignore the word 'merely' and assume that it is constant. This means that the composition of the two bottom squares is a pushout by the second part of Lemma 2. Consequently, the bottom-right square is also a pushout square. Let us write β for the map $\bigvee_C \mathbb{S}^{n+1} \to \bigvee_B \mathbb{S}^{n+1}$ that is described by the middle column of the diagram. We have shown that $C_\beta \simeq X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1}$. Another way to interpret this equivalence is that we gave the space $X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1}$ a good Hurewicz n-connected CW structure V, with $(n+1)$-skeleton $V_{n+1} = \bigvee_B \mathbb{S}^{n+1}$ and attaching map $\alpha_{n+1} = \beta$.

Consider the inclusion $\mathsf{inr} : \bigvee_A \mathbb{S}^{n+1} \to X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1}$. This happens to be a map between CW complexes, so we may approximate it using Theorem 14. Doing so produces a map

$$\mathsf{inr}_{n+1} : \bigvee_A \mathbb{S}^{n+1} \to V_{n+1} = \bigvee_B \mathbb{S}^{n+1},$$

which factors inr as

$$\bigvee_A \mathbb{S}^{n+1} \xrightarrow{\mathsf{inr}_{n+1}} \bigvee_B \mathbb{S}^{n+1} \xrightarrow{\iota_{n+1}} X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1}.$$

Now consider the following diagram.

$$\begin{array}{ccccc} \bigvee_A \mathbb{S}^{n+1} & \xrightarrow{\mathsf{inr}_{n+1}} & \bigvee_B \mathbb{S}^{n+1} & \longrightarrow & X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{1} & \longrightarrow & C_{\mathsf{inr}_{n+1}} & \longrightarrow & X_{n+2} \end{array}$$

The left square is a pushout by definition, and the total square is a pushout for elementary reasons. Thus, the right square is a pushout. Replacing $X_{n+2} \vee \bigvee_A \mathbb{S}^{n+1}$ with C_β, we conclude that X_{n+2} is obtained as the pushout of the span $C_{\mathsf{inr}_{n+1}} \leftarrow \bigvee_B \mathbb{S}^{n+1} \to C_\beta$. An application of the 3×3 lemma tells us that this is equivalent to cofibre of the map $\mathsf{inr}_{n+1} \vee \beta : \bigvee_{A+C} \mathbb{S}^{n+1} \to \bigvee_B \mathbb{S}^{n+1}$. Thus, we have shown that X_{n+2} is of the desired form and we are done. □

Corollary 41 (The Hurewicz Approximation Theorem). *A CW complex is n-connected iff it is Hurewicz n-connected.*

5.2 From homotopy to homology

In order to state our final theorem, we will need the help of the Hurewicz homomorphism. We define it using $\widetilde{H}^{\mathsf{cw}}_n$, but remark that the construction carries over to $\widetilde{H}^{\mathsf{str}}_n$.

Definition 42. *Let X be a CW complex. Define the* ***Hurewicz homomorphism***[5] *$\eta : \pi_n(X) \to \widetilde{H}^{\mathsf{cw}}_n(X)$ on canonical elements $f : \mathbb{S}^n \to_\star X$ by letting $\eta(|f|) : \widetilde{H}^{\mathsf{cw}}_n(X)$ be the image of 1 under the composition $\mathbb{Z} \xrightarrow{\sim} \widetilde{H}^{\mathsf{cw}}_n(\mathbb{S}^n) \xrightarrow{f_*} \widetilde{H}^{\mathsf{cw}}_n(X)$.*

The Hurewicz theorem will provide us with a condition for when this homomorphism is an isomorphism. Before we state and prove it, let us try to understand the groups involved in the 'simple' special case when X is the cofibre C_f of some map of sphere bouquets $f : \bigvee_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n$. This special case will turn out to inform the proof for the general case. As C_f has an explicit CW structure, let us switch our homology theory to $\widetilde{H}^{\mathsf{str}}_n$. Now let us compute $\widetilde{H}^{\mathsf{str}}_n(C_f)$ using the exactness axiom: consider the sequence

$$\bigvee_A \mathbb{S}^n \xrightarrow{f} \bigvee_B \mathbb{S}^n \xrightarrow{\mathsf{cfcod}} C_f \xrightarrow{\mathsf{cfcod}} C_{(\mathsf{cfcod} : \bigvee_B \mathbb{S}^n \to C_f)} \simeq \bigvee_A \mathbb{S}^{n+1}$$

where the final equivalence is the usual characterisation of X_{n+1}/X_n using that C_f has a CW structure. This is a cofibre sequence, and so the following sequence is exact

$$\widetilde{H}^{\mathsf{str}}_n\left(\bigvee_A \mathbb{S}^n\right) \xrightarrow{f_*} \widetilde{H}^{\mathsf{str}}_n\left(\bigvee_B \mathbb{S}^n\right) \xrightarrow{\mathsf{cfcod}_*} \widetilde{H}^{\mathsf{str}}_n(C_f) \to 0 \qquad (1)$$

where the final 0 comes from the fact that $\widetilde{H}^{\mathsf{str}}_n$ vanishes on $\bigvee_A \mathbb{S}^{n+1}$. We can compute the first two homology groups using additivity, and thus we see that $\widetilde{H}^{\mathsf{str}}_n(C_f) \cong \mathbb{Z}[B]/\mathbb{Z}[A]$. Let us now compute the domain of η, i.e. the group $\pi_n(C_f)$.

Proposition 43. *For any $f : \bigvee_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n$ where $n \geq 1$ and A, B : pSet, there is an exact sequence*

$$\pi_n\left(\bigvee_A \mathbb{S}^n\right) \xrightarrow{f_*} \pi_n\left(\bigvee_B \mathbb{S}^n\right) \xrightarrow{\mathsf{cfcod}_*} \pi_n(C_f).$$

Proof sketch. This follows from the Seifert–Van Kampen theorem [19, Example 8.7.17] in the case $n = 1$, and from the Blakers–Massey theorem [9] in the case $n > 1$. □

[5] The fact that this map is a homomorphism boils down to the easy fact that the (group) addition of cellular maps $\mathbb{S}^n \to_\star X$ is again cellular.

We are now almost ready for the Hurewicz theorem. In order to state it, let us define π_n^{ab} to be the abelianisation of the homotopy group functor, i.e. $\pi_n^{\text{ab}}(X) := \pi_n(X)/\text{im}\,[-,-]$ where $[-,-] : \pi_n(X) \times \pi_n(X) \to \pi_n(X)$ is the commutator defined by $[x,y] = xyx^{-1}y^{-1}$. As higher homotopy groups are already abelian, the quotient map $\pi_n(X) \to \pi_n^{\text{ab}}(X)$ is an isomorphism; in what follows, we will simply interpret π_n^{ab} as π_n when $n \geq 2$. We will, with some abuse of notation, view the Hurewicz homomorphism η as being defined over π_n^{ab}. This is justified as the codomain is an abelian group.

Theorem 44 (The Hurewicz Theorem). *The Hurewicz homomorphism $\eta : \pi_n^{\text{ab}}(X) \to \widetilde{H}_n^{\text{cw}}(X)$ is an isomorphism for any $(n-1)$-connected CW complex X.*

Proof. Since we are proving a proposition, we can assume that we have a CW structure X_* and switch our homology theory to $\widetilde{H}_*^{\text{str}}$. Since the map $X_{n+1} \to X_\infty$ is n-connected, the canonical map $\pi_n(X_{n+1}) \to \pi_n(X_\infty)$ is an equivalence. Similarly, $\widetilde{H}_n^{\text{str}}(X_*) = \widetilde{H}_n^{\text{str}}(X_*^{(n+1)})$ by definition. Thus, it suffices to show the theorem for the $(n+1)$-skeleton of X. As X is Hurewicz $(n-1)$-connected, we may assume that $X_n = \bigvee_B \mathbb{S}^n$ and that $X_{n+1} = C_f$ for some $\alpha : \bigvee_A \mathbb{S}^n \to \bigvee_B \mathbb{S}^n$. Elementary algebra tells us that abelianisation is right-exact and thus preserves the exact sequence in Proposition 43. Let us compare this sequence (top sequence below) to the corresponding one for homology in (1) (bottom sequence below).

$$
\begin{array}{ccccc}
\pi_n^{\text{ab}}\left(\bigvee_A \mathbb{S}^n\right) & \xrightarrow{f_*} & \pi_n^{\text{ab}}\left(\bigvee_B \mathbb{S}^n\right) & \xrightarrow{\text{cfcod}_*} & \pi_n^{\text{ab}}(C_f) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\mathbb{Z}[A] & \xrightarrow{\overline{f}} & \mathbb{Z}[B] & \longrightarrow & \mathbb{Z}[B]/\mathbb{Z}[A] \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\widetilde{H}_n^{\text{str}}\left(\left(\bigvee_A \mathbb{S}^n\right)_*\right) & \xrightarrow{f_*} & \widetilde{H}_n^{\text{str}}\left(\left(\bigvee_B \mathbb{S}^n\right)_*\right) & \xrightarrow{\text{cfcod}_*} & \widetilde{H}_n^{\text{str}}\left((C_f)_*\right)
\end{array}
$$

The isomorphisms $\pi_n^{\text{ab}}(\bigvee_C \mathbb{S}^n) \cong \mathbb{Z}[C]$ for $C \in \{A, B\}$ are easily constructed using the Seifert–Van Kampen theorem [19, Example 8.7.17] when $n = 1$ and the Blakers–Massey theorem [9] when $n > 1$. On homology, the isomorphism can be obtained by simply inspecting the related chain complex (alternatively, one can plead to the Eilenberg–Steenrod axioms). The fact that the two left-most squares commute holds almost by definition of the maps involved. Hence we obtain an isomorphism $\pi_n^{\text{ab}}(C_f) \cong \widetilde{H}_n^{\text{str}}((C_f)_*)$. We simply have to verify that this isomorphism is equal to η. It is enough to check this on the inclusion of generators from $\pi_n^{\text{ab}}(\bigvee_B \mathbb{S}^n)$ – but here there is nothing to prove: simply unfolding the definitions involved, it is immediate that the desired equality holds. ◻

We conclude with a special case of this theorem, which we call the *weak Hurewicz theorem*. Although this theorem is a trivial corollary of the full Hurewicz theorem above, we note that it actually can be extracted already from the proof of the full theorem – notably, from parts of the proof that did not use homology at all. The weak Hurewicz theorem inspired the entirely homology free proof of the Serre finiteness theorem by Barton and Campion [1]:

Theorem 45 (The Weak Hurewicz Theorem). *Let C be an $(n-1)$ connected CW complex. There merely exist A, B : pSet s.t. $\pi_n(C) \cong \mathbb{Z}[B]/\mathbb{Z}[A]$.*

6 CONCLUSIONS AND FUTURE WORK

We hope the reader is now convinced that the theory of CW complexes and cellular homology has a home in HoTT. The fact that the results we have proved in this paper – in particular the approximation theorems – are at all provable without any form of choice was initially a surprise to us. The theory of CW complexes and cellular homology as it is developed classically often 'feels' constructive, with many constructions being inductive, but it makes heavy use of choice principles. An important takeaway is that this feeling is justified: a significant part of this theory *is* constructive.

However, the initial motivation behind this project was not to carry out a case study in constructive mathematics. Originally, our development was motivated by the recent proof of the Serre finiteness theorem by Barton and Campion [1]. This proof relies on homology computations and the Hurewicz theorem, thus the formalisation that accompanies this paper should be helpful to the ongoing formalisation of the Serre finiteness theorem (by, in particular, Milner [15]).

This paper also aims to be integrated into a larger project including Mörtberg, which seeks to use cellular (co)homology to reduce homological arguments in HoTT to concrete computations which we can run in proof assistants. The canonical example is the computation of the *Brunerie number* [2], a number whose value is given by a certain cohomology computation which, as it is constructively defined in HoTT, should simply be produced by evaluating it in a proof assistant, but whose evaluation is computationally infeasible. Our hope is that if these computations are ported to a cellular (co)homology theory, where many of them should become feasible, paving the way for *proofs by computation* in HoTT.

It would also be interesting to use our cellular approach to explore more advanced results and constructions such as the Steenrod squares and the (currently open) Künneth formula.

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