Calculating a Brunerie Number

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- But n is still constructively defined. Maybe if we unfold its definition enough, we should be able to deduce $n = \pm 2$ by simply staring at it.

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- In this talk, I will present such a proof

Suspensions

Definition 1 (Suspensions)

The suspension of a type A, denoted ΣA, is given by the following HIT

- $\mathsf{north}, \mathsf{south} : \Sigma A$
- $\mathsf{merid} : A \to \mathsf{north} = \mathsf{south}$

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathsf{merid}"] & \mathbb{1} \arrow[d, "\mathsf{south}"] \\
\mathbb{1} \arrow[r, "\mathsf{north}"'] & \Sigma A \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

Definition 2 (The circle)

We define the circle $\mathbb{S}^1$ by the HIT

- $\mathsf{base} : \mathbb{S}^1$
- $\mathsf{loop} : \mathsf{base} = \mathsf{base}$

Definition 3 (Spheres)

For $n \geq 1$, we define the n-sphere by $(n-1)$-fold suspension of $\mathbb{S}^1$, i.e.

$$\mathbb{S}^n :\equiv \Sigma^{n-1}\mathbb{S}^1$$

Suspension maps

For a pointed type A, there is a canonical map

$$\sigma : A \to \underbrace{\Omega(\Sigma A)}_{:\equiv(\mathsf{north} = \mathsf{north})}$$

given by

$$\sigma(a) = \mathsf{merid}(a) \cdot \mathsf{merid}\,(*_A)^{-1}$$

In particular, when $A = \mathbb{S}^n$, we get

$$\sigma : \mathbb{S}^n \to \Omega\mathbb{S}^{n+1}$$

Joins

Definition 4 (Joins)

The join of two types A and B, denoted $A * B$, is given by

- $\mathsf{inl} : A \to A * B$
- $\mathsf{inr} : B \to A * B$
- $\mathsf{push} : ((a, b) : A \times B) \to \mathsf{inl}(a) = \mathsf{inr}(b)$

$$
\begin{tikzcd}
A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathsf{push}"] & B \arrow[d, "\mathsf{inr}"] \\
A \arrow[r, "\mathsf{inl}"'] & A * B
\end{tikzcd}
$$

Joins

- There is a very useful way to construct maps $A * B \to C$ out of maps $A \times B \to \Omega C$.

Definition 5

Let $f : A \times B \to \Omega C$. Define $\iota_f : A * B \to C$ by

$$\begin{aligned} \iota_f(\mathsf{inl}(a)) &= \star_C \\ \iota_f(\mathsf{inr}(b)) &= \star_C \\ \mathsf{ap}_{\iota_f}(\mathsf{push}(a, b)) &= f(a, b) \end{aligned}$$

- We note that functions $f, g : A \times B \to \Omega C$ can be 'composed':

$$(f \cdot g)(a, b) = f(a, b) \cdot g(a, b)$$

- Q: is there a way of saying that ι is a 'homomorphism' i.e. $\iota_{f \cdot g} = \iota_f + \iota_g$?

An ad hoc construction

- A: yes, if A and B are reasonable.
- Recall, $\pi_n(A) := \|\mathbb{S}^n \to_* A\|_0$

Definition 6

For a pointed type A, define $\pi^*_{n+m+1}(A) = \|\mathbb{S}^n * \mathbb{S}^m \to_* A\|_0$

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Theorem 7

There is a group structure on $\pi^*_{n+m+1}(A)$ such that

- $\pi^*_{n+m+1}(A) \cong \pi_{n+m+1}(A)$
- *For $f, g : \mathbb{S}^n \times \mathbb{S}^m \to \Omega A$, we have $\iota_{f \cdot g} = \iota_f + \iota_g$*

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- **Disclaimer**: Formalisation only for $n = m = 1$ and A 1-connected. (only case we'll use)

$\mathbb{S}^1 * \mathbb{S}^1 \simeq \mathbb{S}^3$

- Here is a particularly important example of the ι-construction.
- There is a canonical map $\smile : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$.
- Composing it with σ gives us $(\sigma \circ \smile) : \mathbb{S}^1 \times \mathbb{S}^1 \to \Omega\mathbb{S}^3$
- Define $\mathcal{F} = \iota_{(\sigma \circ \smile)} : \mathbb{S}^1 * \mathbb{S}^1 \to \mathbb{S}^3$

Proposition 8

$\mathcal{F}$ is an equivalence, and $(\_ \circ \mathcal{F}^{-1}) : \pi_3^*(A) \cong \pi_3(A)$

The Hopf Map and the Brunerie Map

- Define $h, \beta : \mathbb{S}^1 \times \mathbb{S}^1 \to \Omega\mathbb{S}^2$ by

$$h(x, y) = \sigma(y - x)$$
$$\beta(x, y) = \sigma(y) \cdot \sigma(x)$$

- Above, the subtraction comes from the group structure on $\mathbb{S}^1$

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- Above, the subtraction comes from the group structure on $\mathbb{S}^1$
- The induced maps $\iota_h, \iota_\beta : \mathbb{S}^1 * \mathbb{S}^1 \to \mathbb{S}^2$ are called the *Hopf map* and the *Brunerie Map* respectively.

Brunerie's First Theorem

- By precomposition with $\mathcal{F}^{-1} : \mathbb{S}^3 \to \mathbb{S}^2$, we get two corresponding elements $\hat{\iota_h}, \hat{\iota_\beta} : \pi_3(\mathbb{S}^2)$.

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- **Fact:** $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ and is generated by $\hat{\iota_h}$.

Theorem 9 (Brunerie 16)

$\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/n\mathbb{Z}$ *where* n *is the integer s.t.*

$$n \cdot \hat{\iota_h} = \hat{\iota_\beta}$$

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Theorem 9 (Brunerie 16)

$\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/n\mathbb{Z}$ *where n is the integer s.t.*

$$n \cdot \hat{\iota_h} = \hat{\iota_\beta}$$

- We will attempt to solve this equation directly. I claim that $n = -2$ is the solution.

Proof sketch

- In order to show that $n = -2$, we would like to show that

$$\hat{\iota_h} + \hat{\iota_h} = -\hat{\iota_\beta}$$

i.e.

$$(\iota_h \circ \mathcal{F}^{-1}) + (\iota_h \circ \mathcal{F}^{-1}) = -(\iota_\beta \circ \mathcal{F}^{-1})$$

- With our π_3^* construction, the above can be rewritten to something much nicer:

$$(\iota_h + \iota_h) \circ \mathcal{F}^{-1} = (-\iota_\beta) \circ \mathcal{F}^{-1}$$

Proof sketch

- Idea for the rest of the proof: keep rewriting the above equation by passing it through the sequence of isomorphisms

$$\pi_3(\mathbb{S}^2) \xrightarrow{\_\circ\mathcal{F}} \pi_3^*(\mathbb{S}^2) \xrightarrow{(\iota_h\circ\ \_)^{-1}} \pi_3^*(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow{\mathcal{F}\circ\_} \pi_3^*(\mathbb{S}^3)$$

- When we reach $\pi_3^*(\mathbb{S}^2)$, the equation will have turned into something cute!

Step 1

$$\boxed{\pi_3(\mathbb{S}^2) \xrightarrow{\_\circ\mathcal{F}} \pi_3^*(\mathbb{S}^2) \xrightarrow{(\iota_h\circ\_)^{-1}} \pi_3^*(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow{\mathcal{F}\circ\_} \pi_3^*(\mathbb{S}^3)}$$

Applying the highlighted isomorphism above reduces our old equation (in $\pi_3(\mathbb{S}^2)$)

$$(\iota_h + \iota_h) \circ \mathcal{F}^{-1} = (-\iota_\beta) \circ \mathcal{F}^{-1}$$

to the following equation in $\pi_3^*(\mathbb{S}^2)$

$$\iota_h + \iota_h = -\iota_\beta$$

Step 2

$$\boxed{\pi_3(\mathbb{S}^2) \xrightarrow{\_\circ\mathcal{F}} \pi_3^*(\mathbb{S}^2) \xrightarrow{(\iota_h\circ\_)^{-1}} \pi_3^*(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow{\mathcal{F}\circ\_} \pi_3^*(\mathbb{S}^3)}$$

- We would like to rewrite our equation to an equation in $\pi_3^*(\mathbb{S}^1 * \mathbb{S}^1)$ via the highlighted isomorphism.
- To this end, we construct two maps in $f, g : \mathbb{S}^1 * \mathbb{S}^1 \to \mathbb{S}^1 * \mathbb{S}^1$ s.t.

$$\iota_h \circ f = \iota_h + \iota_h$$
$$\iota_h \circ g = \iota_\beta$$

- f is given by $\mathrm{id} + \mathrm{id}$
- g has a somewhat more complicated construction

Step 2

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YOU ARE HERE

- Define $g : \mathbb{S}^1 * \mathbb{S}^1 \to \mathbb{S}^1 * \mathbb{S}^1$ by

$$\begin{aligned} g(\mathsf{inl}(x)) &= \mathsf{inr}(-x) \\ g(\mathsf{inr}(y)) &= \mathsf{inr}(y) \\ \mathsf{ap}_g(\mathsf{push}(x, y)) &= \mathsf{push}(y - x, -x)^{-1} \cdot \mathsf{push}(y - x, y) \end{aligned}$$

- It is very direct to verify that $\iota_h \circ g = \iota_\beta$

Step 3

$$\boxed{\pi_3(\mathbb{S}^2) \xrightarrow{\_\circ\mathcal{F}} \pi_3^*(\mathbb{S}^2) \xrightarrow{(\iota_h\circ\_)^{-1}} \pi_3^*(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow{\mathcal{F}\circ\_} \pi_3^*(\mathbb{S}^3)}$$

YOU ARE HERE

- So we have new equation in $\pi_3^*(\mathbb{S}^1 * \mathbb{S}^1)$:

$$\mathrm{id} + \mathrm{id} = -g$$

- Let's apply the highlighted isomorphism to $(\mathrm{id} + \mathrm{id})$ and g.
- For the LHS: we have, trivially,

$$\mathcal{F} \circ (\mathrm{id} + \mathrm{id}) = \mathcal{F} + \mathcal{F}$$

Step 3

$$\pi_3(\mathbb{S}^2) \xrightarrow{\_\circ\mathcal{F}} \pi_3^*(\mathbb{S}^2) \xrightarrow{(\iota_h\circ\_)^{-1}} \pi_3^*(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow{\mathcal{F}\circ\_} \pi_3^*(\mathbb{S}^3)$$

Proposition 10

$\mathcal{F} \circ g = (-\mathcal{F}) + (-\mathcal{F})$

Proof.

Using the fact that $\mathcal{F}$ is just $\iota_{(\sigma\circ\smile)}$ and the homomorphism property of ι, the proof boils down to proving

$$-((y - x) \smile (-x)) = -(x \smile y)$$
$$(y - x) \smile y = -(x \smile y)$$

which is easy. $\square$

Final step

$$\boxed{\pi_3(\mathbb{S}^2) \xrightarrow{\_\circ\mathcal{F}} \pi_3^*(\mathbb{S}^2) \xrightarrow{(\iota_h\circ\_)^{-1}} \pi_3^*(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow{\mathcal{F}\circ\_} \pi_3^*(\mathbb{S}^3)}$$

- So we are reduced to verifying

$$\mathcal{F} + \mathcal{F} = -((-\mathcal{F}) + (-\mathcal{F}))$$

which, of course, is trivial.

- Combining all the steps, we have shown:

Theorem 11

The Brunerie number (with our definition) is -2.

Concluding remarks

- Paired together with chapters 1–3 in Brunerie's thesis, the above theorem allows us to conclude

Theorem 12

$\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$

- Cool things about this:
  - Much shorter than Brunerie's original proof (skips chapters 4–6)
  - Does not use (co)homology

Concluding remarks

- Ignoring chapters 1–3, we also get a short, standalone proof of the following fact

Theorem 13

If $\pi_4(\mathbb{S}^3)$ is non-trivial, then $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.

- The proof only uses $|n| = 2$, the Freudenthal suspension theorem and Eckmann-Hilton.

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- In particular, an easy corollary is the following:

Theorem 14

If $\Sigma\mathbb{C}P^2 \not\simeq \mathbb{S}^3 \vee \mathbb{S}^5$, then $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.

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- Proving $\Sigma\mathbb{C}P^2 \not\simeq \mathbb{S}^3 \vee \mathbb{S}^5$ can be done using Steenrod squares (WIP, joint with David Wärn)
- But a direct proof, not relying on cohomology would be amazing (suggestions?)

Future work

- Prove $\Sigma\mathbb{C}P^2 \not\simeq \mathbb{S}^3 \vee \mathbb{S}^5$ to complete the new proof of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$
- The Brunerie map is an example of a 'Whitehead product':

$$[\_, \_] : \pi_n(X) \times \pi_m(X) \to \pi_{n+m-1}(X)$$

  These play an important role in the computation of the homotopy groups of spheres. The methods used here could possibly be mimicked for other Whitehead products too.