Formalizing $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ and Computing a Brunerie Number in Cubical Agda

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Abstract—Brunerie's 2016 PhD thesis contains the first synthetic proof in Homotopy Type Theory (HoTT) of the classical result that the fourth homotopy group of the 3-sphere is $\mathbb{Z}/2\mathbb{Z}$. The proof is one of the most impressive pieces of synthetic homotopy theory to date and uses a lot of advanced classical algebraic topology rephrased synthetically. Furthermore, Brunerie's proof is fully constructive and the main result can be reduced to the question of whether a particular "Brunerie number" β can be normalized to ± 2 . The question of whether Brunerie's proof could be formalized in a proof assistant, either by computing this number or by formalizing the pen-and-paper proof, has since remained open. In this paper, we present a complete formalization in the Cubical Agda system, following Brunerie's pen-and-paper proof. We do this by modifying Brunerie's proof so that a key technical result, whose proof Brunerie only sketched in his thesis, can be avoided. We also present a formalization of a new and much simpler proof that β is ± 2 . This formalization provides us with a sequence of simpler Brunerie numbers, one of which normalizes very quickly to -2 in Cubical Agda, resulting in a fully formalized computer assisted proof that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.

I. INTRODUCTION

Homotopy theory originated in algebraic topology, but is by now a central tool in many branches of modern mathematics, such as algebraic geometry and category theory. One of the central notions of study in homotopy theory is that of the homotopy groups of a space X, denoted $\pi_n(X)$. These groups constitute a topological invariant, making them a powerful tool for establishing whether two given spaces can or cannot be homotopy equivalent. The first two such groups of a space are easy to understand: $\pi_0(X)$ characterizes the connected components of X and $\pi_1(X)$ is the fundamental group, i.e. the group of equivalence classes consisting of the loops contained in X up to homotopy. This idea generalizes to higher values of n, for which $\pi_n(X)$ consists of n-dimensional loops up to homotopy. For many spaces, these groups tend to become increasingly esoteric and difficult to compute for large n. This is true also for seemingly tame spaces like spheres, for which $\pi_n(\mathbb{S}^m)$ in general is highly irregular when $n>m\geq 2.1$ This paper concerns the first computer formalization of the classical result that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, a result which is particularly interesting because it gives the whole first stable stem of homotopy groups of spheres, i.e. $\pi_{n+1}(\mathbb{S}^n)$ for $n \geq 3$.

The fact that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ was proved already in the 1930's by Pontryagin using cobordism theory, but we instead follow the synthetic approach to homotopy theory developed in Homotopy Type Theory (HoTT) and popularized by the HoTT Book [2]. In this new approach to homotopy theory, spaces are represented directly as (higher inductive) types and homotopy groups are computed using Voevodsky's univalence axiom [3]. This gives a logical approach to homotopy theory, suitable for computer formalization in proof assistants based on type theory, while also making it possible to interpret results in any suitably structured $(\infty, 1)$ -topos [4].

The basis for our formalization is the 2016 PhD thesis of Brunerie [1] which contains the first synthetic proof in HoTT that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$. The proof is one of the most impressive pieces of synthetic homotopy theory to date and uses advanced machinery from classical algebraic topology developed synthetically, including the symmetric monoidal structure of smash products, (integral) cohomology rings, the Mayer-Vietoris and Gysin sequences, the Hopf invariant, Whitehead products, etc. The formalization of Brunerie's proof has since remained open, primarily due to the highly technical nature of some of the proofs. In this paper, we will present such a formalization in Cubical Agda [5], a *cubical* extension of the Agda proof assistant [6] with native support for computational univalence and higher inductive types (HITs).

In addition to being a very impressive proof in synthetic homotopy theory, Brunerie's proof is particularly interesting as it is fully constructive. The proof consists of two parts, with the first one culminating in Chapter 3 with the definition of a number $\beta: \mathbb{Z}$ such that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$. Since then, this β has been commonly referred to as the *Brunerie number*. Brunerie writes the following about it:

This result is quite remarkable in that even though it is a constructive proof, it is not at all obvious how to actually compute this $[\beta]$. At the time of writing, we still haven't managed to extract its value from its definition. [1, Page 85]

In fact, [1, Appendix B] contains a complete and concise

definition of β as the image of 1 under a sequence of 12 maps:

$$\mathbb{Z} \xrightarrow{} \Omega(\mathbb{S}^1) \xrightarrow{} \Omega^2(\mathbb{S}^2) \xrightarrow{} \Omega^3(\mathbb{S}^3)$$

$$\Omega^3(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow{} \Omega^3(\mathbb{S}^2) \xrightarrow{} \Omega^3(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow{} \Omega^3(\mathbb{S}^3)$$

$$\Omega^2 \|\mathbb{S}^2\|_2 \xrightarrow{} \Omega \|\Omega(\mathbb{S}^2)\|_1 \xrightarrow{} \|\Omega^2(\mathbb{S}^2)\|_0 \xrightarrow{} \Omega(\mathbb{S}^1) \xrightarrow{} \mathbb{Z}$$

By implementing this number in a proof assistant with computational support for univalence and HITs, one should be able to normalize it using a computer to establish that $\beta=\pm 2$ and hence that $\pi_4(\mathbb{S}^3)\cong \mathbb{Z}/2\mathbb{Z}$. In 2016, by the time Brunerie was finishing his thesis, there were some experimental proof assistants based on the cubical type theory of [7], but these were too slow to perform such a complex computation. So, instead of relying on normalization, Brunerie spends the second part of the thesis (Chapters 4–6) to prove, using a lot of the advanced machinery mentioned above, that $|\beta|$ is propositionally equal to 2. However, if one were instead able to compute the number automatically in a proof assistant, this equality would hold definitionally—effectively reducing the complexity and length of the proof by an order of magnitude.

The intriguing possibility of a computer assisted formal proof made many people interested and countless attempts to normalize Brunerie's β have been made using increasingly powerful computers. However, to date, no one has succeeded and it is still unclear whether it is normalizable in a reasonable amount of time. In light of this, it is natural to wonder whether it is possible to simplify Brunerie's number in order to be able to compute it. For example, Brunerie's original definition only involves 1-HITs, as the status of higher HITs was still quite understudied at the time. With a better understanding of higher HITs [8], [9], [10], one quickly sees that the first 3 maps can be combined into one sending 1 to the 3-cell of \mathbb{S}^3 defined as a 3-HIT and not as an iterated suspension as in Brunerie's thesis. Unfortunately, simple optimizations like this do not seem to reduce the complexity of the computation enough and all attempts to run it have thus far failed.

After several unsuccessful attempts at optimizing the computation, we instead decided to formalize the second half of Brunerie's thesis. However, this is by no means straightforward. The first issue appears already in Section 4.1 of Chapter 4, a chapter concerning smash products of spheres. The main result of the section is Proposition 4.1.2, which says that the smash product is a 1-coherent symmetric monoidal product on pointed types. However, the proof of this result is just a sketch and Brunerie writes the following about it:

The following result is the main result of this section even though we essentially admit it. [1, Page 90]

Unfortunately, this result is then used to construct integral cohomology rings, $H^*(X)$, whose cup product, \smile , appears in the definition of the so called Hopf invariant which is crucially used to prove that $|\beta|$ is 2. While one might be convinced that Brunerie's informal proof sketch is correct, it is not obvious how one convinces a proof assistant of this. A

complete formalization would either have to fill in the holes in the proof sketch or find an alternative construction which avoids Proposition 4.1.2. In fact, Brunerie tried very hard to fill these holes using Agda metaprogramming in Agda [11]. However, he never managed to typecheck his computer generated proof of the pentagon identity. Hence, this approach also seems infeasible with current proof assistant technology. Luckily, Brunerie, Ljungström and Mörtberg [12] recently gave an alternative synthetic definition of the cup product on $H^*(X)$ which avoids smash products. This has allowed us to completely skip the problematic Chapter 4 and in particular Proposition 4.1.2, while still following the proof strategy in Chapters 5 and 6. Having a possible strategy for a formal proof, we have then been able to embark on the ambitious project of formalizing Brunerie's proof. Even though we do not need any theory about smash products, there was still a lot left to formalize and our final formalization closely follows Brunerie's proof, except for various smaller simplifications and adjustments which we will discuss in the paper.

In addition to this, we have also formalized a new proof by Ljungström [13] which completely circumvents Chapters 4–6. This major simplification builds on manually calculating the image of the element $\eta: \pi_3(\mathbb{S}^2)$, corresponding to β under the isomorphism $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, by dividing this isomorphism into several maps, tracing η in each step. In particular, the new proof is completely elementary and does not rely on advanced tools such as cohomology. The elements that one obtains while tracing η are all new "Brunerie numbers" that should normalize to ± 2 . In fact, one of these normalizes, in just under 4 seconds on a regular laptop, to -2 in Cubical Agda. Although we still cannot compute Brunerie's original definition, this work can be seen as an alternative solution to Brunerie's conjecture about obtaining a computational proof that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ which relies on simplifying the Brunerie number until it becomes effectively computable.

Outline. The paper closely follows the structure of Brunerie's proof. In Section II, we discuss key results from HoTT that we will need and their formalization in Cubical Agda. Section III, which roughly corresponds to Chapter 2 of Brunerie's thesis, contains some first results on homotopy groups of spheres—e.g. the computation of $\pi_n(\mathbb{S}^m)$ for $n \leq$ m. We then give Brunerie's definition of β and prove that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$, the formalization of which involves the James construction and Whitehead products. The remainder of the paper is then devoted to the formalization of the different proofs that $\beta = \pm 2$. We first discuss the formalization of Chapters 4–6 of Brunerie's proof in Section V. This involves a lot of technical machinery like cohomology, the Hopf invariant, etc. We then, in Section VI, turn our attention to the new elementary proof that $\beta = \pm 2$ and the new Brunerie number which quickly normalizes to -2 in Cubical Agda. We conclude in Section VII with a discussion and comparison of the different formal proofs, as well as some directions for future work.

Formalization. All results in the paper have been formalized

in Cubical Agda and is part of the agda/cubical library (https://github.com/agda/cubical/). The code in the paper is mainly literal Agda code taken verbatim from the library, but we have taken some liberties when typesetting, e.g. shortening notations and omitting some universe levels. A Cubical Agda summary file linking the formalization and paper can be found here. The development typechecks with Agda's --safe flag, which ensures that there are no admitted goals or postulates.

II. HOMOTOPY TYPE THEORY IN Cubical Agda

In this section, we give a concise summary of the key HoTT concepts needed for the proofs and their formalization in Cubical Agda. This roughly corresponds to [1, Chap. 1]. For a more in-depth introduction, see the HoTT Book [2] which also serves as a reference for the formal language "Book HoTT". In this paper, we will present many things with cubical notations, but almost all of the results also hold with minor changes in Book HoTT where paths are represented using Martin-Löf's inductive Id-types [14] instead of cubical path types. In Section VII we will discuss in more detail which proofs crucially rely on cubical features.

All of the results presented in this section were already part of the agda/cubical library before we began our formalization and, while useful as a resource for our notations, experts on HoTT and Cubical Agda can safely skim this section.

A. Elementary HoTT Notions and Cubical Agda Notations

We write $(x:A) \to B$ for dependent function types and denote the identity function by $\mathrm{id}_A:A\to A$. We write $\Sigma_{x:A}(B\,x)$ for the dependent pair type and fst and snd for its projection maps. In what follows, we mean by a *pointed type* a dependent pair (A,\star_A) consisting of a type A and a fixed basepoint $\star_A:A$. For ease of notation, we will often omit the basepoint and simply write A for the pointed type (A,\star_A) . Given two pointed types A and B, the type of *pointed functions* $A\to_\star B$ consists of pairs (f,\star_f) where $f:A\to B$ and $\star_f:f\star_A\equiv\star_B$ witnesses basepoint preservation. Again, we simply write $f:A\to_\star B$ and take \star_f implicit.

HoTT supports inductive types, i.e. types inductively generated by their constructors/points. We write Bool for the type of booleans and $\mathbbm{1}$ for the unit/singleton type with a single point \star_A . A defining feature of HoTT, as opposed to plain Martin-Löf type theory [15], is the existence of *higher inductive types* (HITs). This is a generalization of inductive types where we are not only allowed to specify the generating points of the type in question, but also identifications between these points (and possibly identifications of these identifications, and so on). This is useful for defining quotient types, but also for defining spaces when working in the *types-as-spaces* interpretation of HoTT (see e.g. [2, Table 1] and [16]). Cubical Agda natively supports HITs and a type representing the circle can be defined as follows:

data S^1 : Type where base: S^1 loop: base \equiv base

Here, base \equiv base denotes the type of identifications of base with itself. This is interpreted as the type of paths from base to itself when viewing \mathbb{S}^1 as a space. Hence, the above HIT captures precisely the fact that the circle is a cell complex with one 0-cell (base) and one 1-cell (loop). We always take \mathbb{S}^1 to be pointed by base. In order to discuss the induction principle for \mathbb{S}^1 , we need to discuss paths in more detail. Cubically, paths correspond to functions out of the unit interval, just like in traditional topology. In Cubical Agda, there is a primitive interval type 2 I with endpoints i0 and i1. A path of type $x \equiv y$ between two points x, y : A is a function $p : I \to A$ such that $p : I \to A$ such that

refl:
$$(x : A) \rightarrow x \equiv x$$

refl $x = \lambda i \rightarrow x$

Note that we use "=" for definitional/judgmental equality and "=" for Cubical Agda's path-equality. This can be contrasted with the HoTT Book [2] which uses the opposite convention where "=" is propositional/typal equality.

This type of notational conventions is not the only difference between Cubical Agda and Book HoTT. Many proofs that are complicated in Book HoTT become remarkably direct using the direct treatment of equality using path types. For instance, function extensionality and its inverse funExt⁻ are one-liners [5, Sect. 2.1], while in Book HoTT, this is typically proved as a consequence of the univalence axiom using a rather ingenious proof. Another elementary example of a proof involving _ \equiv _ is cong (called ap in Book HoTT), which applies a function to a path:

cong :
$$(f : A \rightarrow B)$$
 $(p : x \equiv y) \rightarrow f x \equiv f y$ cong $f p \ i = f \ (p \ i)$

Although the treatment of paths in Cubical Agda differs somewhat from Book HoTT, we may still prove path induction: for any dependent type $B:(y:A)\,(p:x\!\equiv\!y)\to \mathsf{Type}$, all dependent functions $f:(y:A)\,(p:x\!\equiv\!y)\to B\,x\,p$ are uniquely determined by $f\,x\,(\mathsf{refl}\,x)$. In Book HoTT, this can be used, among other things, to define the notion of a dependent path, which formalizes the situation when two points a:A and b:B are equal up to a path $p:A\equiv B$. In Cubical Agda, however, the type of dependent paths is primitive:

PathP :
$$(A : I \rightarrow \mathsf{Type}) \rightarrow A \mathsf{i0} \rightarrow A \mathsf{i1} \rightarrow \mathsf{Type}$$

In fact, $_\equiv_$ is just the special case of PathP where the line of paths $(A: I \to \mathsf{Type})$ is constant. We are now ready to describe the induction principle of \mathbb{S}^1 . A dependent function $f: (x: \mathbb{S}^1) \to Bx$ is determined by a point b: B base and a loop $\ell: \mathsf{PathP}(\lambda i \to B(\mathsf{loop}\,i))\,b\,b$. In Cubical Agda, this would be written using pattern matching, as in the left-most definition below, which is introduced side-by-side with the way it would commonly

²For technical reasons, this is actually just a "pre-type" in Cubical Agda.

be written in informal HoTT (such as in Brunerie's thesis):

$$\begin{array}{ll} \mbox{f base} = b & f(\mbox{base}) = b \\ \mbox{f (loop } i) = \ell \ i & \mbox{ap}_f(\mbox{loop}) = \ell \end{array}$$

B. More Higher Inductive Types

Let us now introduce the remaining HITs used in [1]. These come equipped with induction principles analogous to that of \mathbb{S}^1 . To define higher spheres, we need suspensions:

```
data Susp (A : Type) : Type where north : Susp <math>A south : Susp A merid : A \rightarrow north \equiv south
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We always take suspensions to be pointed by north. We may now define the n-sphere, for $n \geq 1$, by $\mathbb{S}^n = \operatorname{Susp}^{n-1} \mathbb{S}^1$ where $\operatorname{Susp}^{n-1}$ denotes (n-1)-fold suspension. We also define $\mathbb{S}^{-1} = \bot$ (the empty type) and $\mathbb{S}^0 = \operatorname{Bool}$. We remark that we could equivalently have defined \mathbb{S}^1 as the suspension of \mathbb{S}^0 as is done in [1]. Our reason for not doing so is that certain functions using \mathbb{S}^1 appear to compute better with the base/loop definition. Furthermore, this is the definition used in already existing code in the agda/cubical library.

We may also capture the (homotopy) pushout of a span $B \stackrel{f}{\leftarrow} A \stackrel{g}{\rightarrow} C$ by the following HIT:

```
data Pushout (f: A \rightarrow B) (g: A \rightarrow C): Type where inl: B \rightarrow Pushout f g inr: C \rightarrow Pushout f g push: (a: A) \rightarrow inl (f a) \equiv inr (g a)
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We use pushouts to define the wedge sum of two pointed types, denoted $A \vee B$, the join of two types, denoted $A \star B$, and the cofiber of a map $f: A \to B$, denoted cofib f:

Two particularly important functions out of wedge sums are:

$$\begin{array}{lll} \nabla:A\vee A\to A & i^\vee:A\vee B\to A\times B \\ \nabla\;(\operatorname{inl}\;x)=x & i^\vee\;(\operatorname{inl}\;a)=(a\;,\;\star_B) \\ \nabla\;(\operatorname{inr}\;x)=x & i^\vee\;(\operatorname{inr}\;b)=(\star_A\;,\;b) \\ \nabla\;(\operatorname{push}\;\star_1\;i)=\star_A & i^\vee\;(\operatorname{push}\;\star_1\;i)=(\star_A\;,\;\star_B) \end{array}$$

C. Truncation levels and n-truncations

An important concept in HoTT is Voevodsky's h-levels [17], which gives rise to the notion of an n-type. Since types in HoTT are interpreted as spaces (or rather, as homotopy types), they are not only determined by their points but also by which higher paths they may contain. We say that a type A is an n-type if all (n+1)-dimensional structure of A is trivial. Formally, this is captured by an inductive definition. We say that A is a (-2)-type if it is contractible, i.e. consisting of a single point, as captured by is $\operatorname{Contr} A = \Sigma_{a_0:A}((a:A) \to a_0 \equiv a)$.

We inductively say that A an (n+1)-type if for any x, y : A, the type $x \equiv y$ is an n-type. We call (-1)-types propositions and 0-types sets.

We can turn any type A into an n-type by n-truncation, denoted $\|A\|_n$. We often use direct definitions of (-1)- and 0-truncation in our formalization, and similar constructions work for any fixed value of n, but not when n is arbitrary. For higher n we rely on the hub-and-spoke construction [2, Sect. 7.3]. One caveat with truncations is that a map $f:A \to B$ does not, in general, induce a map $f:\|A\|_n \to B$. This is, however, the case when B is an n-type. In particular, f always induces a function $\|f\|_n:\|A\|_n \to \|B\|_n$.

D. Univalence, loop spaces, and h-spaces

In order to introduce Voevodsky's univalence principle [3], we need to define the (homotopy) fiber of a function. Given a function $f:A\to B$ and a point b:B, we define the fiber of f over b by fib f $b=\Sigma_{x:A}(f\ a\equiv b)$. We say that $f:A\to B$ is an equivalence, written $f:A\simeq B$, if fib f b is contractible for all b:B. In order to prove that a function $f:A\to B$ is an equivalence, it suffices to provide an inverse $f^-:B\to A$ and two paths $f\circ f^-\equiv \operatorname{id}_B$ and $f^-\circ f\equiv \operatorname{id}_A$. If f is also pointed, we write $f:A\simeq_\star B$.

Univalence states that the canonical map $A \equiv B \to A \simeq B$, defined by path induction, is an equivalence. In particular, we get a map $ua:A\simeq B\to A\equiv B$ promoting equivalences to paths. This provides us with a useful method for transferring proofs between equivalent types.

Transferring proofs is, however, not the only use case of univalence in HoTT. It can also be used to characterize *loop spaces* of HITs. This is often done using the *encode-decode method* [2, Sect. 8.1.4], a type theoretic analogue of proofs by contractibility of total spaces of fibrations. In HoTT, we define the loop space of a pointed type A, by $\Omega A = (\star_A \equiv \star_A)$. This is again pointed by refl \star_A , so we may iterate this definition to get the nth loop space of A, denoted $\Omega^n A$. Loop spaces belong to a particularly important class of types called h-spaces. These consist of a pointed type B equipped with a unital magma structure

$$\mu: B \times B \to B$$

$$\mu_l: (b:B) \to \mu(\star_B, B) \equiv b$$

$$\mu_r: (b:B) \to \mu(b, \star_B) \equiv b$$

satisfying $\mu_l \star_B \equiv \mu_r \star_B$. Another particularly important h-space for our purposes is \mathbb{S}^1 , for which we will use + to denote its binary operation. \mathbb{S}^1 also comes equipped with a notion of inversion which we will denote by -. In fact, \mathbb{S}^1 is a commutative and associative h-space.

III. FIRST RESULTS ON HOMOTOPY GROUPS OF SPHERES

In this section, we cover [1, Chap. 2], which introduces some elementary results on the homotopy groups of spheres. All of these results can also be found in [2]. Before even stating them, we need homotopy groups:

Definition 1 (Homotopy groups). For $n : \mathbb{N}$, we define the nth homotopy group of a pointed type A by:

$$\pi_n(A) = \| \mathbb{S}^n \to_{\star} A \|_0$$

The name homotopy group should be taken with a grain of salt: it, in general, only has a group structure when $n \geq 1$ (abelian when $n \geq 2$). The structure may be defined by considering the equivalence $(\mathbb{S}^n \to_{\star} A) \simeq (\mathbb{S}^{n-1} \to_{\star} \Omega A)$, where the latter type has a multiplication given by pointwise path composition. An alternative definition of $\pi_n(A)$ is via loop spaces. There is an equivalence $\omega_n : \Omega^n A \simeq (\mathbb{S}^n \to_{\star} A)$ and, hence, we could equivalently have defined $\pi_n(A)$ by setting $\pi_n(A) = \|\Omega^n A\|_0$. This makes the group structure on $\pi_n(A)$ more transparent: it is simply path composition. This is the definition used in [2]. Brunerie uses both definitions and often passes between the two without comment.

An elementary but crucial result for the computation of homotopy groups is the existence of the *long exact sequence* of homotopy groups. Its proof is usually phrased using the loop space definition of homotopy groups [2, Theorem 8.4.6]. For ease of notation, let us simply write fib f for the fiber of a pointed function $f: A \to_{\star} B$ over the basepoint of B.

Proposition 1 (LES of homotopy groups). For any pointed map $f: A \to_{\star} B$, there is a long exact sequence

$$\pi_n(\mathsf{fib}\,f) \xrightarrow{\longleftarrow} \pi_n(A) \xrightarrow{\longrightarrow} \pi_n(B)$$

$$\pi_{n-1}(\mathsf{fib}\,f) \xrightarrow{\longleftarrow} \dots$$

When analyzing loop spaces and homotopy groups of suspensions, the following function is of great importance. It will be used in many constructions to come.

Definition 2 (The suspension map). Given a pointed type A, there is a canonical map $\sigma: A \to \Omega$ (Susp A) given by

$$\sigma x = \operatorname{merid} x \cdot (\operatorname{merid} \star_A)^{-1}$$

This induces a homomorphism on homotopy groups by post-composition:

$$\pi_n(A) \xrightarrow{\sigma_*} \pi_n(\Omega\left(\operatorname{Susp} A\right)) \xrightarrow{\cong} \pi_{n+1}(\operatorname{Susp} A)$$

We will often, with some abuse of notation, simply write σ_* for this composition. We also define $\sigma_n:\|A\|_n\to\Omega\|\operatorname{Susp} A\|_{n+1}$ by

$$\sigma_n |x| = \operatorname{cong} |\underline{\hspace{0.1cm}}| (\sigma x)$$

We will soon see the suspension map in action, but first we need the following elementary result.

Proposition 2 (Join of spheres). $\mathbb{S}^n * \mathbb{S}^m \simeq \mathbb{S}^{n+m+1}$.

Proof: The statement is easily proved by induction, using $\mathbb{S}^{n+1} \simeq \operatorname{Susp} \mathbb{S}^n$ and $(\operatorname{Susp} A) * B \simeq \operatorname{Susp} (A * B)$.

In particular, Proposition 2 gives us an equivalence $\mathbb{S}^1 \star \mathbb{S}^1 \simeq \mathbb{S}^3$. Using this fact, we define the following map, which will play a crucial role in the analysis of $\pi_4(\mathbb{S}^3)$.

Definition 3 (Hopf map). We define hopf : $\mathbb{S}^3 \to \mathbb{S}^2$ by the composition $\mathbb{S}^3 \xrightarrow{\sim} \mathbb{S}^1 \star \mathbb{S}^1 \xrightarrow{h} \mathbb{S}^2$ where h is given by

```
h: \mathbb{S}^1 \star \mathbb{S}^1 \to \mathbb{S}^2
h (inl x) = north
h (inr y) = north
h (push (x, y) i) = \sigma(y - x) i
```

It turns out that the following is true [2, Theorem 8.5.1].

Proposition 3 (The fiber of the Hopf map). *The fiber of* hopf is equivalent to \mathbb{S}^1 , i.e. fib hopf $\simeq \mathbb{S}^1$.

Proposition 3 gives us a fibration sequence $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$ which, in particular, will allow us to connect homotopy groups of \mathbb{S}^2 with those of \mathbb{S}^3 and \mathbb{S}^1 . For this, we need to introduce the notion of *connectedness*. We say that a type A is n-connected if $\|A\|_n$ is contractible. Similarly, we say that a function $f:A\to B$ is n-connected if all of its fibers are n-connected. This means, in particular, that the induced function $\|f\|_n:\|A\|_n\to\|B\|_n$ is an equivalence. The following is an immediate consequence of the definition of n-truncations.

Lemma 1 (Connectedness of spheres). For $n \ge -1$, \mathbb{S}^n is (n-1)-connected.

Using Lemma 1, we can easily prove the following:

Proposition 4 ([1, Prop. 2.4.1]). For n < m, the group $\pi_n(\mathbb{S}^m)$ is trivial.

For the sake of coherence, let us take the liberty of mentioning some results from [1, Chap. 3] already here, since they also concern low-dimensional homotopy groups of spheres. A crucial result is the following theorem [2, Theorem 8.6.4]:

Theorem 1 (Freudenthal suspension theorem). Given an n-connected and pointed type A, the map $\sigma: A \to \Omega$ (Susp A) is 2n-connected.

On can easily deduce from Theorem 1 that, in particular, $\sigma_n:\|A\|_n\to\|\Omega\left(\operatorname{Susp} A\right)\|_n$ is an equivalence. This allows us to prove the following result:

Corollary 1. For $n \geq 1$, we have $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$. Furthermore, $\pi_n(\mathbb{S}^n)$ is generated by $i_n = |\operatorname{id}_{\mathbb{S}^n}|$.

Proof: The synthetic proof of the classical result that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ is due to Licata and Shulman [18]. The fact that $\pi_2(\mathbb{S}^2) \cong \pi_1(\mathbb{S}^1)$ is given by the LES associated to the Hopf fibration combined with Proposition 4. The fact that $\pi_{n+1}(\mathbb{S}^{n+1}) \cong \pi_n(\mathbb{S}^n)$ is an immediate consequence of Theorem 1. The second statement follows by induction on n.

We have now analyzed all homotopy groups $\pi_n(\mathbb{S}^m)$ with $n \leq m$. This yields the following:

Proposition 5. Post-composition by hopf induces an isomorphism $\pi_3(\mathbb{S}^3) \cong \pi_3(\mathbb{S}^2)$.

Proof: By Propositions 1 and 3, we get an exact sequence

$$\pi_3(\mathbb{S}^1) \to \pi_3(\mathbb{S}^3) \xrightarrow{\mathsf{hopf}_*} \pi_3(\mathbb{S}^2) \to \pi_2(\mathbb{S}^1)$$

as $\pi_n(\mathbb{S}^1)$ vanishes for n > 1, hopf_{*} is an isomorphism.

Corollary 2. There is an isomorphism $\psi : \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$. Furthermore, $\pi_3(\mathbb{S}^2)$ is generated by hopf.

Proof: By Corollary 1 we know that $\pi_3(\mathbb{S}^3)$ is generated by the identity function on \mathbb{S}^3 . We know that the isomorphism $\pi_3(\mathbb{S}^3) \cong \pi_3(\mathbb{S}^2)$ is given by post-composition by hopf and thus the generator of $\pi_3(\mathbb{S}^3)$, is mapped to hopf.

A. Formalization of Chapter 2

Most results were already added to agda/cubical by Mörtberg & Pujet [19], Ljungström [20], and Brunerie, Ljungström & Mörtberg [12]. The Freudenthal suspension theorem was formalized in Cubical Agda by Evan Cavallo [21], using a direct cubical proof following [2, Thm. 8.6.4]. Corollary 1 was given a direct proof, following the computation of cohomology groups of spheres in [12].

There were some technical difficulties related to the equivalence $\omega_n:\Omega^nA\simeq(\mathbb{S}^n\to_\star A)$, which is used to show that the two different definitions of homotopy groups are equivalent. In several proofs, it is more natural to work on the left-hand-side of ω_n . At the same time, working on the right-hand-side often makes constructing elements easier (compare, for instance, an explicit description of the generator of $i_3:\pi_3(\mathbb{S}^3)$ described as a 3-loop in \mathbb{S}^3 to the very compact definition $i_3=|\mathrm{id}_{\mathbb{S}^3}|$). This means that we often have to translate between the two definitions. One particularly important example is the LES of homotopy groups associated to a function $A\to_\star B$. On each level, the maps are given as follows:

$$\Omega^n \text{ (fib } f) \xrightarrow{\Omega^n \text{ fst}} \Omega^n A \xrightarrow{\Omega^n f} \Omega^n B$$

This is then transported to the definition of homotopy groups as maps from spheres via ω_n . For the proof of e.g. Corollary 2, we need to know that the maps in the sequence are given as follows:

$$\pi_n(\operatorname{fib} f) \xrightarrow{\operatorname{fst}_*} \pi_n(A) \xrightarrow{f_*} \pi_n(B)$$

What we need is then more than just an equivalence $\omega_n:\Omega^nA\simeq(\mathbb{S}^n\to_\star A)$ – we need to show that this equivalence is functorial. This is implicitly assumed in Brunerie's thesis, but, in Cubical Agda, we need to make it precise. Formalizing this fact is not entirely trivial. First, we need a tractable definition of the equivalence in question. It can be described inductively with base case $\omega_1:\Omega A\to(\mathbb{S}^1\to_\star A)$ given by:

$$\omega_1 p \operatorname{\mathsf{base}} = \star_A$$
 $\omega_1 p (\operatorname{\mathsf{loop}} i) = p i$

which we take to be pointed by refl. It is easy to verify that this is an equivalence. We define ω_{n+1} by the composition:

$$\begin{split} \Omega^{n+1}\,A &= \Omega\left(\Omega^n\,A\right) \xrightarrow{\quad \Omega\,\omega_n \quad} \Omega\left(\mathbb{S}^n \to_\star A\right) \\ &\xrightarrow{\quad \text{funExt}_\star^- \quad} \left(\mathbb{S}^n \to_\star \Omega\,A\right) \\ &\xrightarrow{\quad \ } \left(\mathbb{S}^{n+1} \to_\star A\right) \end{split}$$

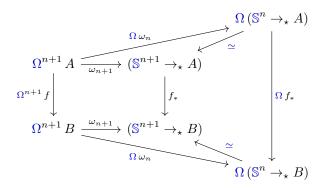
where the last arrow comes from the adjunction Susp $\dashv \Omega$. This is a composition of equivalences, and hence an equivalence. We then need to verify that the following commutes

$$\Omega^{n} A \xrightarrow{\omega_{n}} (\mathbb{S}^{n} \to_{\star} A)$$

$$\Omega^{n} f \downarrow \qquad \qquad \downarrow f_{*}$$

$$\Omega^{n} B \xrightarrow{\omega_{n}} (\mathbb{S}^{n} \to_{\star} B)$$

This can be proved inductively. The base case is easy and the inductive step is given by the following diagram



where the commutativity of the outer square comes from the base case paired with the inductive hypothesis, the triangles from the definition of ω_{n+1} and the right-most square from a straightforward argument.

IV. THE BRUNERIE NUMBER

Here we give an overview of the first half of Brunerie's proof. This corresponds to [1, Chap. 3] and culminates in the isomorphism $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$ for an at this point unknown "Brunerie number" $\beta : \mathbb{Z}$. We also discuss the formalization of this part of the proof and various simplifications to it found during the formalization.

A. The James Construction

To define β , Brunerie uses the *James construction* [22], which he introduced in HoTT and partially formalized in [23].

Proposition 6 (James construction). For a $(k \ge 0)$ -connected pointed type A, there are types J_n A with inclusions

$$\mathsf{J}_0\:A \overset{j_0}{\longleftrightarrow} \mathsf{J}_1\:A \overset{j_1}{\longleftrightarrow} \mathsf{J}_2\:A \overset{j_2}{\longleftrightarrow} \cdots$$

such that its sequential colimit $J_{\infty} A \simeq \Omega$ (Susp A). Furthermore, $j_n : J_n A \hookrightarrow J_{n+1} A$ is (n(k+1)+(k-1))-connected.

A consequence of Proposition 6 is the following fact

Proposition 7. Given a $(k \ge 0)$ -connected type A, there is a (3k+1)-connected map $J_2 A \to \Omega$ (Susp A).

The proof of Proposition 7 uses that $J_{\infty} A$, the sequential colimit of the sequence in Proposition 6, can be shown to be equivalent to $\Omega(\operatorname{Susp} A)$. This, paired with some results on the connectivity of sequential colimits, gives the statement.

Theorem 2.
$$\pi_4(\mathbb{S}^3) \cong \pi_3(\mathsf{J}_2\,\mathbb{S}^2)$$

Proof: Because \mathbb{S}^2 is 1-connected, Proposition 7 tells us that there is a 4-connected map

$$\mathsf{J}_2\,\mathbb{S}^2\to\Omega\,(\mathsf{Susp}\,\mathbb{S}^2)=\Omega\,(\mathbb{S}^3)$$

In particular, it is 3-connected and induces an equivalence $\| J_2 \mathbb{S}^2 \|_3 \simeq \| \Omega \mathbb{S}^3 \|_3$. We get:

$$\pi_4(\mathbb{S}^3) \cong \pi_3(\Omega \mathbb{S}^3) \cong \pi_3(\mathsf{J}_2 \mathbb{S}^2)$$

B. Formalization of the James Construction

This is a particularly technical part of Brunerie's thesis, primarily due to the high number of higher coherences which need to be verified in the proof of Proposition 6. While this has, subsequent to our efforts, been formalized in its entirety by Rongji [24], we have taken a shortcut by giving a direct proof of Theorem 2, which means we do not in fact need the full James construction. Consequently, we instead give direct definitions of $J_n A$ for $n \le 2$ for a pointed type A.

Definition 4 (Low dimensional James construction). We define $J_0 A = \mathbb{1}$ and $J_1 A = A$. The type $J_2 A$ is defined as the pushout of the span $A \times A \stackrel{i^{\vee}}{\leftarrow} A \vee A \stackrel{\nabla}{\rightarrow} A$.

We remark that the construction in Definition 4 is not definitionally the same as Brunerie's; in his thesis, these constructions are theorems rather than definitions. Here we take them as definitions. With $J_n A$ defined this way, the map $j_0: J_0 A \rightarrow J_1 A$ is just the constant pointed map and $j_1: J_1 A \rightarrow J_2 A$ is inr.

Before we continue, let us temporarily redefine S^2 to be the following equivalent HIT. This will make some of the following constructions more compact.

data S^2 : Type where base: S^2 surf: refl base \equiv refl base

The next lemma will be crucial. It is a special case of the *Wedge Connectivity Lemma* [2, Lemma 8.6.2], of which we have formalized a version of the proof of the sphere case in [12, Lemma 8]. From the point of view of formalization, this proof is easier to work with since it gives more useful definitional equalities.

Lemma 2 (Wedge connectivity for \mathbb{S}^2). Let $P: \mathbb{S}^2 \times \mathbb{S}^2 \to 2$ -Type. Any function $f: (x: \mathbb{S}^2 \times \mathbb{S}^2) \to Px$ is induced by the following data:

$$f_l: (x:\mathbb{S}^2) \to P(x, \mathsf{base})$$

 $f_r: (y:\mathbb{S}^2) \to P(\mathsf{base}, y)$
 $f_{lr}: f_l \, \mathsf{base} \equiv f_r \, \mathsf{base}$

Before we discuss the formalization of Theorem 2 stated with the low dimensional James construction, we first construct the following function. The goal is to define a family of equivalences $f_x : \| J_2 S^2 \|_3 \simeq \| J_2 S^2 \|_3$ over $x : S^2$. We

do this by truncation elimination and pattern matching on x, starting with the base case:

$$f_{\mathsf{base}} \, |\operatorname{inl}\left(x,y
ight)| = |\operatorname{inl}\left(x,y
ight)|$$

$$f_{\mathsf{base}} \, |\operatorname{inr}z| = |\operatorname{inl}\left(\mathsf{base}\,,z
ight)|$$

We omit the path constructors, which are all easy coherences. It is an easy lemma that f_{base} is equal to the identity on $\|\mathbf{J}_2\|_2^2\|_3$. To complete the definition of f_x , we need to consider the case when x = surf i j. This amounts to providing a dependent function:

$$f_{\mathsf{surf}}: (x: \| \mathsf{J}_2 \mathbb{S}^2 \|_3) \to \Omega^2 (\| \mathsf{J}_2 \mathbb{S}^2 \|_3, f_{\mathsf{base}} x)$$

To do this, we will, in particular, need to provide a family of fillers $Q_{(x\,,\,y)}: \mathsf{refl}_{|\,\mathsf{inl}\,(x\,,\,y)\,|} \equiv \mathsf{refl}_{|\,\mathsf{inl}\,(x\,,\,y)\,|}$. This is a 1-type, and thus Lemma 2 applies. We define:

$$Q_{(\mathsf{base}\,,\,y)}\,i\,j = |\inf(\mathsf{surf}\,i\,j\,,y)|$$

$$Q_{(x\,,\,\mathsf{base})}\,i\,j = |\inf(x\,,\,\mathsf{surf}\,i\,j)|$$

The fact that these two constructions agree when both x and y are base is a technical but relatively straightforward lemma. Thereby, $Q_{(x,y)}$ is defined. We may now define $f_{\sf surf}$:

$$f_{\mathsf{surf}} \, | \, \mathsf{inl} \, (x \, , y) \, | = Q_{(x \, , \, y)}$$

$$f_{\mathsf{surf}} \, | \, \mathsf{inr} \, z \, | = Q_{(\mathsf{base} \, , \, z)}$$

The higher cases are easy due to the fact that the goal becomes 0-truncated, making it sufficient to define them for base : \mathbb{S}^2 . Thus, f_x is defined for all $x : \mathbb{S}^2$.

Lemma 3. For $x : \mathbb{S}^2$, f_x is an automorphism on $\| J_2 \mathbb{S}^2 \|_3$.

Proof: To make coming proofs easier, this is proved by explicitly constructing the inverse analogously to f_x .

We are now ready to prove the following statement, which is a rephrasing of Theorem 2.

Proposition 8.
$$\Omega \parallel \mathbb{S}^3 \parallel_4 \simeq \parallel \mathsf{J}_2 \mathbb{S}^2 \parallel_3$$

Proof: We take $\mathbb{S}^3 = \operatorname{Susp} \mathbb{S}^2$, where \mathbb{S}^2 is defined using base/surf as above. We employ the encode-decode method and define Code : $\|\mathbb{S}^3\|_4 \to 3$ -Type. Since the universe of 3-types is a 4-type, we may do so by truncation elimination, letting Code | north | = Code | south | = $\|\mathbf{J}_2\mathbb{S}^2\|_3$ and Code | merid xi | = $\operatorname{ua} f_x i$. We now need to define a family of functions $\operatorname{decode}_x : \operatorname{Code} x \to |\operatorname{north}| \equiv x$ over $x: \|\mathbb{S}^3\|_4$. The key step is defining $\operatorname{decode}_{|\operatorname{north}|} : \|\mathbf{J}_2\mathbb{S}^2\|_3 \to \Omega \|\mathbb{S}^3\|_4$. On point constructors, it is given by:

$$\begin{split} \mathsf{decode}_{\mid \, \mathsf{north} \, \mid} \left(\mathsf{inl} \left(x \, , y \right) \right) &= \sigma \, x \cdot \sigma \, y \\ \mathsf{decode}_{\mid \, \mathsf{north} \, \mid} \left(\mathsf{inr} \, z \right) &= \sigma \, z \end{split}$$

which is easily verified to be coherent with the higher constructors. At this point, we may follow the usual encodedecode heuristic [2, Section 8.9] to prove that decode north is an equivalence in a technical but direct manner.

We get Theorem 2 as an immediate corollary of Proposition 8 via the same sequence of isomorphisms as in the proof of Theorem 2.

C. Definition of the Brunerie Number

Brunerie's goal is now to analyze $\pi_3(J_2 \mathbb{S}^2)$. The first result needed is the following:

Definition 5 (Whitehead map). Given two pointed types A and B, there is a map:

W:
$$A * B \rightarrow \text{Susp } A \vee \text{Susp } B$$

W (inl a) = inr north
W (inr b) = inl north
W (push $(a, b) i$) =
(cong inr $(\sigma b) \cdot \text{push } \star_1 \overset{-1}{\longrightarrow} \cdot \text{cong inl } (\sigma a) i$

For our purposes, we only need the case when $A = B = \mathbb{S}^1$ (although all of the following results appear in full generality in Brunerie's thesis). We get a composite map:

$$\mathbb{S}^3 \xrightarrow{\simeq} \mathbb{S}^1 \star \mathbb{S}^1 \xrightarrow{\mathsf{W}} \mathbb{S}^2 \vee \mathbb{S}^2$$

This induces, via pre-composition, a Whitehead product:

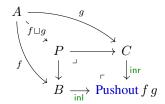
$$\pi_2(\mathbb{S}^2) \times \pi_2(\mathbb{S}^2) \xrightarrow{[-,-]} \pi_3(\mathbb{S}^2)$$

Recall that we denote by i_2 the generator of $\pi_2(\mathbb{S}^2)$. Brunerie shows, in particular, the following about its relation to the Whitehead product.

Theorem 3. The kernel of the suspension map $\sigma_* : \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$ is generated by $[i_2, i_2]$.

The key technical component in the proof is the *Blakers-Massey Theorem*, first formalized in HoTT by Favonia, Finster, Licata & Lumsdaine in [25]:

Theorem 4 (Blakers-Massey). Consider the diagram

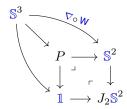


where P is the pullback along in and inr, i.e. $P = \sum_{(b,c):B\times C} (\operatorname{inl} b \equiv \operatorname{inr} c)$, and $f\sqcup g$ is defined by

$$(f \sqcup g) a = (f a, g a, \operatorname{push} a)$$

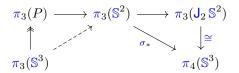
If f and g are n- respectively m-connected, then $f \sqcup g$ is (n+m)-connected.

Theorem 3 is proved by considering the following diagram



Verifying that the outer square is a pushout square is technical and we omit the proof here. Above, P is simply the fiber of

inr : $\mathbb{S}^2 \to J_2 \mathbb{S}^2$. The leftmost map is 2-connected since \mathbb{S}^3 is 2-connected and the top map is 0-connected since \mathbb{S}^3 and \mathbb{S}^2 are both 1-connected. Consequently, by Theorem 4, we get that the map $\mathbb{S}^3 \to P$ is 2-connected and thus induces a surjection on π_3 . We get the following diagram:



where the sequence on the top comes from the long exact sequence of homotopy groups associated to P. The dashed map sends the generator $i_3:\pi_3(\mathbb{S}^3)$ to $[i_2,i_2]:\pi_3(\mathbb{S}^2)$ by definition.

Theorem 3 motivates the following definition. Recall that we denote by ψ the isomorphism $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$.

Definition 6 (Brunerie number). We define the Brunerie number $\beta : \mathbb{Z}$ by $\beta = \psi [i_2, i_2]$.

We may now state the main result of [1, Chap. 3].

Corollary 3.
$$\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$$
.

Proof: The statement follows immediately from Theorem 3 and the fact that $\sigma_*: \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$ is surjective, which is a consequence of Theorem 1.

D. Formalization of the definition of the Brunerie number

The formalization of this part was straightforward. Arguably the most technical result, the Blakers-Massey theorem, was already available in the library thanks to Rongji [26]. Most of the remaining results were essentially just diagram chases which, in a proof assistant, can be somewhat technical. Most work went into verifying that $J_2 \mathbb{S}^2$ is the cofiber of $\nabla \circ W$, the proof of which followed Brunerie's closely.

In this section we found the only obvious mistake in Brunerie's thesis. On page 82, in his definition of the push-case for W, the path component in the middle was not inverted, making the term ill-typed. Naturally, this was of no mathematical significance and something Brunerie immediately would have noticed if he would have attempted to provide a computer formalization of this construction.

V. Brunerie's Proof that $|\beta| \equiv 2$

This section concerns the final three Chapters (4–6) of Brunerie's thesis. The main goal here is proving that $|\beta| \equiv 2$.

We will not discuss Chapter 4 in much detail. Chapter 4 is devoted to smash products and, in particular, their symmetric monoidal structure. Brunerie used this in subsequent chapters to define and prove properties about the *cup product*, a graded multiplicative operation on cohomology groups which will be used to show that $|\beta| \equiv 2$. This chapter has turned out to be incredibly difficult to formalize due to the large number of higher coherences involved in the proofs [11].

Luckily, it turns out that Chapter 4 can be avoided altogether and that this in fact makes some difficult proofs later on very direct. For this reason, the results in Chapter 4 were omitted completely from our formalization. The reason for this is that all results regarding smash products in Brunerie's thesis concern, in some way, pointed maps out of smash products. In this case, we may exploit the adjunction of maps out of smash products and bi-pointed maps:

$$(A \wedge B \rightarrow_{\star} C) \simeq (A \rightarrow_{\star} (B \rightarrow_{\star} C))$$

Here, $B \to_{\star} C$ is taken to be pointed by the constant map. As shown in [12], it is arguably easier to define the cup product on the right-hand side of the adjunction, which effectively means that we never have to work with smash products when formalizing cohomology theory.

A. Cohomology and the Hopf Invariant

[1, Chap. 5] introduces integral cohomology groups and rings, and gives a construction of the Mayer-Vietoris sequence. In more detail, Brunerie defines the integral Eilenberg-MacLane spaces by $K_0 = \mathbb{Z}$ and $K_n = \| \mathbb{S}^n \|_n$ for $n \ge 1$. This allows for a definition of the (integral) cohomology of X:

$$\mathsf{H}^n(X) = \| X \to \mathsf{K}_n \|_0$$

The fact that $\Omega \mathsf{K}_{n+1} \simeq \mathsf{K}_n$ follows by a proof completely analogous to that of Corollary 1. Brunerie uses this equivalence to carry over the (commutative) h-space structure on $\Omega \mathsf{K}_{n+1}$ to that of K_n . This provides a notion of addition $+_k : \mathsf{K}_n \times \mathsf{K}_n \to \mathsf{K}_n$ which lifts to $\mathsf{H}^n(X)$ by post-composition, thereby endowing $\mathsf{H}^n(X)$ with a group structure.

In this chapter, Brunerie provides a synthetic construction of the Mayer-Vietoris sequence, i.e. the long exact sequence

$$\mathsf{H}^0(D) \xrightarrow{\qquad \qquad \mathsf{H}^0(B) \times \mathsf{H}^0(C) \xrightarrow{\qquad \qquad } \mathsf{H}^0(A)$$

$$\mathsf{H}^1(D) \xrightarrow{\qquad \qquad } \dots$$

where D denotes the pushout of a span $B \xleftarrow{f} A \xrightarrow{g} C$. A direct application gives us, for $n \ge 1$, that $\mathsf{H}^n(\mathbb{S}^m) \cong \mathbb{Z}$ if n = m and $\mathsf{H}^n(\mathbb{S}^m) \cong \mathbb{I}$ otherwise. This gives, by another application of the sequence, the following result:

Lemma 4. For any $f: \mathbb{S}^3 \to \mathbb{S}^2$ we have

$$\mathsf{H}^n(\mathsf{cofib}\, f) \cong \left\{ egin{array}{ll} \mathbb{Z} & n \in \{0,2,4\} \\ \mathbb{1} & \mathsf{otherwise} \end{array} \right.$$

Let us briefly fix $f: \mathbb{S}^3 \to \mathbb{S}^2$. Denote by γ_2 and γ_4 the generators of $H^2(\operatorname{cofib} f)$ and $H^4(\operatorname{cofib} f)$ respectively given by the image of $1: \mathbb{Z}$ under the isomorphism in Lemma 4. These generators may be used to define an invariant on $\mathbb{S}^3 \to \mathbb{S}^2$ called the *Hopf Invariant*. This is done as follows:

Definition 7 (Hopf Invariant). The Hopf Invariant of f is the unique integer $H | f : \mathbb{Z}$ such that $\gamma_2 \smile \gamma_2 \equiv H | f \cdot \gamma_4$.

We remark that the above definition is given for the more general class of maps $\mathbb{S}^{2n-1} \to \mathbb{S}^n$ in Brunerie's thesis. For our purposes, the above special case suffices. In particular, we may see HI as a function $\pi_3(\mathbb{S}^2) \to \mathbb{Z}$. The following turns out to be true:

Proposition 9. HI is a homomorphism $\pi_3(\mathbb{S}^2) \to \mathbb{Z}$.

Proof: We first rephrase $f + g : \pi_3(\mathbb{S}^2)$ as a composition

$$\mathbb{S}^3 \to \mathbb{S}^3 \vee \mathbb{S}^3 \xrightarrow{f \vee g} \mathbb{S}^2$$

By analyzing the cohomology of $\operatorname{cofib}(f \vee g)$ and the action on generators of the obvious maps from $\operatorname{cofib}(f \vee g)$, $\operatorname{cofib} f$ and $\operatorname{cofib} g$ into $\operatorname{cofib}(f+g)$, one arrives at the result with some elementary algebra.

Finally, the Hopf invariant of our element of interest $[i_2, i_2]$ is computed (up to a sign), using an argument similar to that of the proof of Proposition 9.

Proposition 10. $|HI[i_2,i_2]| \equiv 2$

We are now almost done: if there is an element $f: \pi_3(\mathbb{S}^2)$ such that $\mathsf{HI} f \equiv 1$, then HI is an isomorphism $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$. Since isomorphisms of this type are unique up to a sign, Proposition 10 tells us that also for the standard isomorphism $\psi: \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, we must have $|\psi[i_2, i_2]| \equiv 2$, i.e. $|\beta| \equiv 2$. Hence, we have so far shown the following:

Lemma 5. If
$$\text{HI } f \equiv 1$$
 for some $f : \pi_3(\mathbb{S}^2)$, then $|\beta| \equiv 2$.

The final chapter of Brunerie's thesis is devoted to proving the antecedent of Lemma 5.

B. Formalization of Cohomology and the Hopf Invariant

This section was largely covered by Brunerie, Ljungström and Mörtberg in [12] and thus also available in agda/cubical. Hence, what remained to be formalized in Chapter 5 was the Hopf invariant and Propositions 9 and 10. The formalization of these propositions was straightforward and we were able to translate Brunerie's proofs in a direct manner. This is not surprising as the proofs are very algebraic.

For simplicity, we only formalized these propositions as they stand here and not their generalizations to higher spheres (i.e. as in [1, Prop. 5.4.3 & 5.4.4]). We remark, however, that the formalized proofs easily should be rephrasable for the general Hopf invariant of maps $\mathbb{S}^{2n-1} \to \mathbb{S}^n$.

C. The Gysin sequence

This section corresponds to [1, Chap. 6]. In order to be able to apply Lemma 5, this chapter is devoted to proving that $|\mathsf{HI}| \mathsf{hopf}| \equiv 1$, where, recall, $\mathsf{hopf}: \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf map—the generator of $\pi_3(\mathbb{S}^2)$ from Definition 3. This amounts to analyzing the cup product on the cohomology of cofib hopf. It is well-known that cofib hopf gives a model of the complex projective plane $\mathbb{C}P^2$ (see e.g. [27, Example 4.45]), so let us simply write $\mathbb{C}P^2$ from now on. In order to show that $|\mathsf{HI}| \mathsf{hopf}| \equiv 1$, it suffices to show that $- \smile \gamma_2 : \mathsf{H}^2(\mathbb{C}P^2) \to \mathsf{H}^4(\mathbb{C}P^2)$ is an isomorphism for $\gamma_2 : \mathsf{H}^2(\mathbb{C}P^2)$ a generator. Brunerie does this by constructing the Gysin sequence synthetically.

Proposition 11 (Gysin sequence). Let B be a pointed and 0-connected type and $P: B \to \mathsf{Type}$ be a fibration with $P \star_B \simeq_\star \mathbb{S}^{n-1}$. Let $E = \Sigma_{b:B}(Pb)$ be the total space of P. If

there is a family of maps $c:(b:B) \to (\operatorname{Susp}(Pb) \to_{\star} \mathsf{K}_n)$ with c_{\star_B} a generator of $\mathsf{H}^n(\mathbb{S}^n)$, then there is an element $e_n:\mathsf{H}^n(B)$ and a long exact sequence

Moreover, c (and also e_n) exists when B is 1-connected.

In order to make use of this, we need the following result.

Proposition 12. There is a fibration $P: \mathbb{C}P^2 \to \mathsf{Type}$ with $P \star_{\mathbb{C}P^2} \simeq_{\star} \mathbb{S}^1$ and total space \mathbb{S}^5 .

Proposition 12 is a special case of the following result.

Proposition 13 (Iterated Hopf Construction). Given an associative h-space A, let $h_A: A*A \to \operatorname{Susp} A$ denote the associated Hopf map. There is a fibration $\operatorname{cofib} h_A \to \operatorname{Type}$ with fiber A and total space A*A*A.

We consider the particular case when $A = \mathbb{S}^1$ in Proposition 13. In this case, the map $h_{\mathbb{S}^1}: \mathbb{S}^1 \star \mathbb{S}^1 \to \mathbb{S}^2$ corresponds to the usual Hopf map under the equivalence $\mathbb{S}^1 \star \mathbb{S}^1 \simeq \mathbb{S}^3$ and hence $\operatorname{cofib} h_{\mathbb{S}^1} \simeq \mathbb{C}P^2$. The total space of this is $\mathbb{S}^1 \star \mathbb{S}^1 \star \mathbb{S}^1$ which is equivalent to \mathbb{S}^5 by Proposition 2 and thus we have proved Proposition 12. The associated Gysin sequence gives us the main result of this section:

Proposition 14. $|H|h| \equiv 1$

Proof: Since $\mathbb{C}P^2$ is 1-connected, Proposition 11 combined with Proposition 12 gives us an element $e_2: H^2(\mathbb{C}P^2)$ and a sequence

$$\mathsf{H}^{i-1}\big(\mathbb{S}^5\big)\to\mathsf{H}^{i-2}\big(\mathbb{C}P^2\big)\xrightarrow{-\ \smile\ e_2}\mathsf{H}^i\big(\mathbb{C}P^2\big)\to\mathsf{H}^i\big(\mathbb{S}^5\big)$$

When $1 \le i \le 4$, $\operatorname{H}^i(\mathbb{S}^5)$ vanishes. Setting i = 2, we get that e_2 must be a generator of $\operatorname{H}^2(\mathbb{C}P^2)$, and thus equal to the generator $\gamma_2 : \operatorname{H}^2(\mathbb{C}P^2)$ up to a sign. Setting i = 4, we get that $- \smile e_2$ must be an isomorphism of groups $\operatorname{H}^2(\mathbb{C}P^2) \cong \operatorname{H}^4(\mathbb{C}P^2)$ and hence $e_2 \smile e_2$ is a generator. Consequently, so is $\gamma_2 \smile \gamma_2$, and thus $|\operatorname{HI}| \operatorname{hopf} = 1$.

Proposition 14 combined with Lemma 5 gives the desired path: $|\beta| \equiv 2$. This completes Brunerie's proof and Corollary 3 gives us the main result:

Theorem 5.
$$\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$$

D. Formalization of the Gysin Sequence

Formalizing the results from Chapter 6 was more challenging, but was greatly aided by the alternative construction of the cup product discussed above. The first technical lemma, which is crucial for the construction of the Gysin sequence is:

Lemma 6. Given $x : K_n$ and $y : K_m$, we have

$$\operatorname{cong}\;(\lambda\,a\to a\smile_k y)\;(\sigma_n\,x)\,{\equiv}\,\sigma_{n+m}(x\smile_k y)$$

In Brunerie's thesis, this lemma relies on a result which in turn requires the symmetric monoidal structure of the smash product (in particular, it uses the *pentagon identity*). With the alternative construction of the cup product, however, this result follows immediately from the definition of the cup product.

Lemma 6 is used to show that the map

$$g^i: \mathsf{K}_i \to (\mathbb{S}^n \to_{\star} \mathsf{K}_{i+n})$$

 $g^i x = \lambda y \to x \smile_k \iota y$

is an equivalence, which is crucially used in the construction of the Gysin sequence. Above, $\iota:\mathbb{S}^n\to \mathsf{K}_n$ is a generator of $\mathsf{H}^n(\mathbb{S}^n)$. For reference, g^i is the map $g^i_{\star B}$ in the proof of [1, Prop. 6.1.2]. While the general idea of Brunerie's proof of this statement is correct, it was difficult to formalize directly. The primary reason for this is that Brunerie does not pay much attention to the fact that the objects of interest are not just functions, but *pointed* functions. Fortunately for us, the whole proof is very direct with the alternative definition of the cup product. Formalizing Brunerie's proof with pointedness of functions respected would have been hard, especially without machinery external to [1] (e.g. [12, A.2, Lemma 27]).

After these subtleties were dealt with, the formalization of the Gysin sequence could proceed following Brunerie's proof closely. In our initial formalization, we made a slight adjustment to the indexing of the Gysin sequence. This removed some bureaucracy but happened at the cost of generality.³ This made verifying that Proposition 14 slightly less direct, because we no longer had access to the case

$$\mathsf{H}^1(\mathbb{S}^5) \to \mathsf{H}^0(\mathbb{C}P^2) \xrightarrow{-\smile e_2} \mathsf{H}^2(\mathbb{C}P^2) \to \mathsf{H}^2(\mathbb{S}^5)$$

which is used by Brunerie to show that the element e_2 : $H^2(\mathbb{C}P^2)$, for which $-\smile e_2: H^2(\mathbb{C}P^2) \to H^4(\mathbb{C}P^2)$ is an isomorphism, is indeed a generator. However, in practice, this is not a big problem. In fact, it provides a nice example of a proof by computation. It is very direct to manually show that the map $i: \mathbb{C}P^2 \to K_2$ induced by $i(\inf x) = |x|$ is equal to the underlying map of e_2 . The fact that i generates $H^2(\mathbb{C}P^2)$ can then be verified by computation: applying the isomorphism $H^2(\mathbb{C}P^2) \cong \mathbb{Z}$ to |i| returns 1 by normalization in Cubical Agda. We stress, for those skeptical of this method, that it also is very direct to provide a "manual" formalization of this fact.

The final step of the formalization was Proposition 13, i.e. the iterated Hopf construction. Although technical, the formalization could be carried out following Brunerie closely.

VI. THE SIMPLIFIED NEW PROOF AND NORMALIZATION OF A BRUNERIE NUMBER

It turns out that not only Chapter 4, but also Chapters 5–6 can be avoided. As conjectured by Brunerie, it would be possible to do this by simply normalizing the Brunerie number. While we still cannot normalize his original definition of it, Ljungström showed in [13] that we can at least provide a computation of a substantially simplified Brunerie number.

 $^{^3\}mathrm{A}$ more general form of the Gysin sequence using Brunerie's indexing has later been added to agda/cubical.

This is defined via a more tractable description of the isomorphism $\pi_3(\mathbb{S}^2)\cong\mathbb{Z}$ as a composition of simpler isomorphisms, relying on an alternative definition of π_3 in terms of $\mathbb{S}^1\star\mathbb{S}^1$. The idea is then to trace $[i_2,i_2]:\pi_3(\mathbb{S}^2)$ step by step through these isomorphisms. This gives a sequence of new Brunerie numbers and one of these quite surprisingly normalizes to -2 in Cubical Agda in a matter of seconds.

The trick to give a more tractable definition of $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ is to redefine the third homotopy group of a type A as $\pi_3^*(A) = \|\mathbb{S}^1 \star \mathbb{S}^1 \to_{\star} A\|_0$. This reformulation of π_3 can be given an explicit group structure, such that pre-composition by $\mathbb{S}^1 \star \mathbb{S}^1 \simeq \mathbb{S}^3$ induces an isomorphism $\pi_3(A) \cong \pi_3^*(A)$. We briefly outline the proof by first defining the following product:

$$_ \smile_1 _ : \mathbb{S}^1 \to \mathbb{S}^1 \to \mathbb{S}^2$$

base $\smile_1 y = \text{north}$
loop $i \smile_1 y = \sigma y i$

In fact, \smile_1 behaves like a "cup product" on \mathbb{S}^1 :

Proposition 15. For $x, y : \mathbb{S}^1$, we have

$$x \smile_1 y \equiv -(y \smile_1 x)$$

$$x \smile_1 \mathsf{base} \equiv \mathsf{base}$$

$$x \smile_1 (x+y) \equiv x \smile_1 y$$

where - denotes inversion on \mathbb{S}^2 .

Proof: All three identities are direct by pattern matching on x and, for the first one, also on y. The first one uses, in the case where x and y are both loops, an easy lemma which states for any 2-loop $p: \Omega^2 A$ we have that $p^{-1} \equiv (\lambda i j \rightarrow p j i)$.

This operation plays an important role in the definition of the equivalence $\mathbb{S}^1 \times \mathbb{S}^1 \simeq \mathbb{S}^3$.

Proposition 16. The following map is an equivalence:

F:
$$\mathbb{S}^1 \star \mathbb{S}^1 \to \mathbb{S}^3$$

F (inl x) = north
F (inr y) = north
F (push (x, y) i) = σ $(x \smile_1 y)$ i

We omit the proof which is essentially just technical pathalgebra. The fact that F uses \smile_1 , which satisfies the laws in Proposition 15, lets us analyze $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ in a more algebraic manner. We now redefine $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ via the following decomposition, primarily defined in terms of post- and precomposition with F and its inverse:

Definition 8. Let $\theta : \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ be defined by the following sequence of isomorphisms

where h is the map from Definition 3 and the last map can be chosen to be any reasonable description of the isomorphism $\xi : \pi_3(\mathbb{S}^3) \cong \mathbb{Z}$ sending i_3 to 1.

The goal is to trace the image of $[i_2, i_2] : \pi_3(\mathbb{S}^2)$ under θ . Let us define the following three underlying functions of elements $\eta_1 : \pi_3^*(\mathbb{S}^2), \ \eta_2 : \pi_3^*(\mathbb{S}^1 \star \mathbb{S}^1)$ and $\eta_3 : \pi_3^*(\mathbb{S}^3)$:

```
\begin{array}{l} \eta_1\text{-fun}: \mathbb{S}^1 \star \mathbb{S}^1 \to \mathbb{S}^2 \\ \eta_1\text{-fun} & (\operatorname{inl} x) = \operatorname{north} \\ \eta_1\text{-fun} & (\operatorname{inr} y) = \operatorname{north} \\ \eta_1\text{-fun} & (\operatorname{push} (x \, , \, y) \, i) = (\sigma \, y \cdot \sigma \, x) \, i \\ \\ \eta_2\text{-fun} & (\operatorname{push} (x \, , \, y) \, i) = (\sigma \, y \cdot \sigma \, x) \, i \\ \\ \eta_2\text{-fun} & (\operatorname{inl} x) = \operatorname{inr} (-x) \\ \eta_2\text{-fun} & (\operatorname{inr} y) = \operatorname{inr} y \\ \eta_2\text{-fun} & (\operatorname{push} (x \, , \, y) \, i) = \\ & (\operatorname{push} & (y - x \, , \, -x) \, ^{-1} \cdot \operatorname{push} & (y - x \, , \, y)) \, i \\ \\ \eta_3\text{-fun} & (\operatorname{inl} x) = \operatorname{north} \\ \eta_3\text{-fun} & (\operatorname{inr} y) = \operatorname{north} \\ \eta_3\text{-fun} & (\operatorname{push} & (x \, , \, y) \, i) = \\ & (\sigma & (x \smile_1 y) \, ^{-1} \cdot \sigma & (x \smile_1 y) \, ^{-1}) \, i \end{array}
```

The claim is now that the image of $[i_2, i_2]$ under the chain of isomorphisms can be described as follows:

$$[i_2,i_2] \overset{\mathsf{F}^*}{\longmapsto} \eta_1 \overset{(\mathsf{h}_*)^{-1}}{\longmapsto} \eta_2$$

$$\downarrow \mathsf{F}_* \qquad \downarrow \mathsf{F}_*$$

$$\eta_3 \overset{(\mathsf{F}^{-1})^*}{\longmapsto} (-2)i_3 \overset{\xi}{\longmapsto} \pm 2$$

Lemma 7. $F^*[i_2, i_2] \simeq \eta_1$

Proof: The definition of η_1 matches that of $|\nabla \circ \mathbf{W}|$: $\pi_3^*(\mathbb{S}^2)$, and so the statement holds by construction of the Whitehead product.

Lemma 8.
$$(h_*)^{-1} \eta_1 \equiv \eta_2$$

Proof: Applying h_* on both sides gives the equation $\eta_1 \equiv h_* \eta_2$. The underlying functions of these elements agree definitionally on inl and inr, and the push-case reduces to a simple application of the laws described in Proposition 15.

Lemma 9.
$$(F^{-1})^* \eta_2 \equiv \eta_3$$

Proof: The proof is similar to that of Lemma 8.

Theorem 6.
$$\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$$

Proof: By uniqueness (up to a sign) of isomorphisms $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, it suffices, according to Corollary 3, to show that the image of $[i_2, i_2]$ under θ is ± 2 . That is:

$$(\xi \circ (\mathsf{F}^{-1})^* \circ \mathsf{F}_* \circ (\mathsf{h}_*)^{-1} \circ \mathsf{F}^*)[i_2, i_2] \equiv \pm 2$$

By Lemmas 7 to 9, it suffices to show that

$$(\xi \circ (\mathsf{F}^{-1})^*) \eta_3 \equiv \pm 2$$

One can easily show that $\mathsf{F}^{-1} \eta_3 \equiv (-2) i_3$, and hence we get

$$(\xi \circ (\mathsf{F}^{-1})^*) \eta_3 \equiv (-2) (\xi i_3) \equiv -2$$

In addition to providing a new and much shorter proof of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, this gives us a sequence of new Brunerie numbers, $\beta_1, \beta_2, \beta_3 : \mathbb{Z}$, of decreasing complexity:

$$\begin{split} \beta_1 &= (\xi \circ (\mathsf{F}^{-1})^* \circ \mathsf{F}_* \circ (\mathsf{h}_*)^{-1}) \, \eta_1 \\ \beta_2 &= (\xi \circ (\mathsf{F}^{-1})^* \circ \mathsf{F}_*) \, \eta_2 \\ \beta_3 &= (\xi \circ (\mathsf{F}^{-1})^*) \, \eta_3 \end{split}$$

This gives new hope for Brunerie's conjecture about a proof by normalization. This may be captured as follows:

Theorem 7 (New Brunerie numbers). If either of $\beta_1, \beta_2, \beta_3$: \mathbb{Z} normalizes to ± 2 , then $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.

Ideally, we could normalize β_1 . This, however, turns out to be difficult, as it does not bypass the main hurdle of computing the inverse of the isomorphism $\pi_3^*(\mathbb{S}^2) \cong \pi_3^*(\mathbb{S}^{1*}\mathbb{S}^1)$ induced by the Hopf map, which has a rather indirect construction coming from the LES of homotopy groups associated to the Hopf fibration. This problem does not apply to β_2 , for which the computation does not rely on the problematic inverse. Unfortunately, also β_2 fails to normalize in reasonable time in Cubical Agda. This is surprising, as the only maps playing a fundamental role here are two applications of the equivalence $\mathbb{S}^1*\mathbb{S}^1\simeq\mathbb{S}^3$, which is not too involved, and one application of ξ which may be compactly described via

$$\pi_3(\mathbb{S}^3) \xrightarrow{|\square|_*} \mathsf{H}^3(\mathbb{S}^3) \xrightarrow{\cong} \mathbb{Z}$$

and computes relatively well if the last isomorphism is constructed as in [12].⁴ We have hence, at the time of writing, not been able to normalize even β_2 , despite many optimizations of the functions involved. We are, however, able to normalize β_3 after some minor modifications to η_3 and the map $\pi_3^*(\mathbb{S}^3) \to \mathbb{Z}$. This optimized version of β_3 , normalizes to -2 in Cubical Agda in just under 4 seconds, thereby giving us an at least partially computer assisted proof of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.

We emphasize again that, β_2 is a vastly simplified version of β since the isomorphism $\pi_3(\mathbb{S}^2) \cong \pi_3(\mathbb{S}^3)$ never has to be computed. Hence, it is rather surprising that computations break down already at this stage. This tells us that Cubical Agda has a long way to go before any direct computation of the original β is feasible. We hope that this could be useful for benchmarking in future optimizations of Cubical Agda.

Finally, we address the elephant in the room: why is there a minus sign popping up? In other words, have we really chosen the, in some way, canonical isomorphism? The isomorphism $\pi_3(\mathbb{S}^3) \cong \mathbb{Z}$ maps, as expected, i_3 to 1, so it can hardly be the culprit. Neither can the equivalence $F: \mathbb{S}^1 \star \mathbb{S}^1 \simeq \mathbb{S}^3$, since it is applied equally in the constructions of hopf and of $[i_2, i_2]$. We could, however, have defined the push-case for h by

h (push
$$(x, y)$$
 $i) = \sigma (x - y) i$

in which case $\theta[i_2, i_2]$ would have been sent to 2 and hopf to 1 (note that this is only possible since altering h would alter

the definition of θ). The construction of h that we have given is, however, precisely the one which fell out by unfolding our formalization Brunerie's construction of the corresponding map. If this indeed is what Brunerie intended, we may also conclude that the original Brunerie number β is equal to -2. We stress that this merely is a fun fact and of no mathematical importance to Brunerie's proof or our formalization.

VII. CONCLUSION

In this paper, we have presented three formalizations of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ in the Cubical Agda system. For the different proofs that $|\beta| \equiv 2$, the line count is roughly as follows:

- 1) Brunerie's original proof [$\sim 9,000 \text{ LOC}$]
- 2) Ljungström's direct calculation of β [~ 600 LOC]
- 3) A computer assisted reformulation of 2) [\sim 400 LOC] As always, the number of lines of code (LOC) should be taken with a grain of salt. First, the 9,000 LOC in the first formalization exclude over 8,000 LOC from [26], [21], [12]. Second, these numbers also exclude many elementary results used in the formalization. We also stress that these line counts exclude \sim 9,000 LOC for Chapters 1-3 which are relevant to all three proofs.

Formalization 1), which constituted the bulk of this paper, was a formalization of Brunerie's pen-and-paper proof, taking some convenient shortcuts when possible. The problem of formalizing Brunerie's proof has been a widely discussed open problem in HoTT/UF, and we hope that our efforts here provide a satisfactory solution to it. Formalizations 2) and 3) were of Ljungström's simplified calculation of the Brunerie number, β , as presented in [13]. The very similar proofs 2) and 3) differ in that 3) uses Cubical Agda to carry out part of the computation of the new Brunerie number automatically. Perhaps equally important, we have seen that 3) provides us with new Brunerie numbers $\beta_1, \beta_2 : \mathbb{Z}$ which are far simpler than the original one, but still do not normalize in a reasonable amount of time. Our hope is that these can prove useful in future optimizations of Cubical Agda and related systems, as they could help shed some light on where the normalization of the original Brunerie number breaks down.

We remark that proofs 1) and 2) could be done in Book HoTT and do not use any cubical machinery in a fundamental way, making them interpretable in any suitably structured $(\infty, 1)$ -topos [4]. We hence claim that, in our formalizations, we do not crucially rely on computations using univalence and HITs to prove anything that we could not have proved by hand in Book HoTT. Nevertheless, the Cubical Agda system has been very helpful in the formalization, primarily due to its native support for HITs and definitional computation rules for higher constructors. Formalization 3), however, is only valid in a system with computational support for univalence as it crucially relies on normalization of proof terms involving univalence. It would be interesting to run this in other cubical systems, like cubicaltt [28], redtt [29], cooltt [30], etc.

We also remark that our formalization of Brunerie's proof does not cover all results of Brunerie's thesis in full generality. For instance, we have not developed his proof concerning

⁴As noted in [20], the Freudenthal suspension theorem should be avoided here as it has a tendency to lead to very slow computations. This is another way in which we deviate from Brunerie's β .

Whitehead products in full generality. We leave this generalization for future work. This would tie in nicely with another possible direction of future research, namely that of investigating whether the approach outlined in Section VI can be used to compute other Whitehead products. In addition, describing their graded quasi-Lie algebra structure is work in progress. Another related project is the proof of the symmetric monoidal structure on smash products, i.e. the main result of [1, Chap. 4]. While this would not make the proof of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ easier, it would be interesting to see whether Brunerie's proof could actually be formalized, in the way that he intended it. Naturally, this question is also interesting on its own right. Also this is ongoing work and some of the biggest hurdles have recently been overcome.

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