Some properties of Whitehead products

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Whitehead products are graded operations on homotopy groups which play an important role in homotopy theory. In HoTT, they have been around since Brunerie's thesis [Bru16] where they were used to, for instance, define the Brunerie number. More recently, they have been used to characterise H-space structures by Buchholtz et al. [Buc+23] and to construct a kind of EHP sequence by Cagne et al. [Cag+24]. Despite their central role in many arguments, little is known about their properties in HoTT. Here, I will prove their fundamental Lie algebra properties and, in particular, bilinearity – a property which is crucially used in upublished work by Tom Jack and myself attempting to compute the second stable homotopy group of spheres and which is mentioned by Cagne et al. as a component in the computation of $\pi_1(S^2 \to S^2, f)$.

Some notation and elementary machinery All types in this note are taken to be pointed and adopt the notation $\star_A : A$ for basepoints. We will need three HITs: suspensions (ΣX), joins ($X * Y$), wedge sums ($X \vee Y$) and smash products ($X \wedge Y$). These will, respectively, be pushouts of the spans

$$\mathbb{1} \leftarrow A \rightarrow \mathbb{1} \qquad X \leftarrow X \times Y \rightarrow Y \qquad X \leftarrow \mathbb{1} \rightarrow Y \qquad \mathbb{1} \leftarrow X \vee Y \rightarrow X \times Y$$

We rename the constructors of ΣA to the usual north, south and merid (and take north to be the basepoint). We will capture the canonical loops in ΣA by $\sigma : X \to \Omega\Sigma X$ defined by $\sigma(x) := \mathsf{merid}\,(x) \cdot \mathsf{merid}\,(\star_X)^{-1}$. For a function $f : \Sigma X \to_\star Y$, we use σ to define $\widetilde{f}_{(-)} : X \to_\star \Omega Y$, letting $\widetilde{f}_{(-)} = \Omega(f) \circ \sigma$. It will be an important fact that both suspensions [BH18, Proposition 5.3] and joins [LM23, §6] are co-H-spaces (i.e. the type of functions out of these types are H-spaces). We will always use $+$ to denote the binary operation on $A \to_\star B$ where A is a co-H-space.

Defining 'the' Whitehead product Although the Whitehead product was originally defined as a multiplication on homotopy groups, it is possible to generalise it. Instead of defining the Whitehead product of two maps with spheres as domains, we can define it for maps whose suspensions are domains. This is called the *generalised Whitehead product* [Ark62]. It also appears in [Bru16].

Definition 1. *Let $f : \Sigma X \to_\star Z$ and $g : \Sigma Y \to_\star Z$. We define their Whitehead product $[f, g] : X * Y \to_\star Z$ by*

$$[f, g]\,(\mathsf{inl}(x)) := \star_Z \qquad [f, g]\,(\mathsf{inr}(y)) := \star_Z \qquad \mathsf{ap}_{[f,g]}(\mathsf{push}\,(x, y)) := \widetilde{g}_y \cdot \widetilde{f}_x$$

We obtain the usual Whitehead product (on homotopy groups) by letting $X := \mathbb{S}^n$ and $Y := \mathbb{S}^m$ and using that $\mathbb{S}^n * \mathbb{S}^m \simeq \mathbb{S}^{n+m+1}$.

Since $X * Y \simeq \Sigma(X \wedge Y)$, it also makes sense to define the above Whitehead product to be of type $\Sigma(X \wedge Y) \to_\star Z$. Let us introduce different notation for this product by writing $[f, g]_s : \Sigma(X \wedge Y) \to_\star Z$. This construction sends the point constructors to $\star_Z$ and a canonical pair $\langle x, y\rangle : X \wedge Y$ to the commutator $\widetilde{f}_x^{-1}\widetilde{g}_y\widetilde{f}_x\widetilde{g}_y^{-1}$. These two definitions are good for different things. While the first one is somewhat more flexible when manipulating it directly, the second one makes sure that we always have a suspension in the domain, making iterations of Whitehead products well-typed.

Super Lie algebra structure The main result we prove here is that Whitehead products form a super Lie algebra. Concretely, this means that the Whitehead product $[-, -] : \pi_n(X) \times \pi_m(X) \to \pi_{n+m-1}(X)$ is bilinear, graded commutative, i.e. $[f, g] = (-1)^{nm}[g, f]$ for $f : \pi_n(X)$ and $g : \pi_m(X)$, and satisfies the Jacobi identity, i.e. $(-1)^{nk}[f, [g, h]] + (-1)^{mn}[g, [h, f]] + (-1)^{km}[h, [f, g]] = 0$ for $f : \pi_n(X)$, $g : \pi_m(X)$ and $h : \pi_k(X)$. Here we have implicity taken $n, m, k > 1$ in order to guarantee that our homotopy groups are abelian.

Proving these properties on the level of homotopy groups gets somewhat messy. It would be nicer if we could prove them on the level of generalised Whitehead products. This is not very hard. In what follows, X, Y and Z are all suspensions, so e.g. $X \simeq \Sigma X'$ where X' is pointed. We also remark that we will overload the $+$-symbol and use it for the H-space structure both on $\Sigma A \to B$ and $A * B \to C$.

- **Left linearity** is expressed by asking that for $f, g : \Sigma X \to_\star Z$ and $h : \Sigma Y \to_\star Z$, we have that $[f + g, h] = [f, h] + [g, h]$.
- **Right linearity** is expressed in the analogous way.
- **Symmetry** is expressed by asking that for any $f : \Sigma X \to Z$ and $g : \Sigma Y \to_\star Z$, we have that $[f, g] = [g, f] \circ \mathsf{swap}_{X,Y}$ where $\mathsf{swap}_{X,Y} : X * Y \to Y * X$.
- **The Jacobi identity** is most conveniently expressed using the smash version of Whitehead products: we are asking that for any $f : \Sigma X \to_\star W$, $g : \Sigma Y \to_\star W$ and $h : \Sigma Z \to_\star W$, we have that $[f, [g, h]_s]_s = [[f, g]_s, h]_s \circ e_0 + [g, [f, h]_s]_s \circ e_1$ where e_0 and e_1 are the obvious correction equivalences making the expression well-typed.

Although they capture rather different aspects of Whitehead products, all properties can be proved using the exact same proof technique: by solving word problems. Let us show left linearity here and leave it at that – the same idea can be used to show the remaining properties.

Proposition 1 (Left linearity)**.** *Let $f, g : \Sigma X \to Z$ and $h : \Sigma Y \to Z$ with $X \simeq_\star \Sigma X'$ for some pointed type X'. The Whitehead product is left linear, i.e. $[f + g, h] = [f, h] + [g, h]$.*

Proof. Let us try the simplest proof possible – a direct construction of the desired equality by pattern matching. Explicitly, what we need to provide is

- for every $x : X$, a path $p_x : \star_Z = \star_Z$ (showing that the functions agree on inl),
- for every $y : Y$, a path $q_y : \star_Z = \star_Z$ (showing that the functions agree on inr),
- for every $x : X$ and $y : Y$, a filler of the square

$$\begin{array}{ccc} \star_Z & \xrightarrow{(\widetilde{f}_x^{-1}\widetilde{h}_y\widetilde{f}_x\widetilde{h}_y^{-1})(\widetilde{g}_x^{-1}\widetilde{h}_y\widetilde{g}_x\widetilde{h}_y^{-1})} & \star_Z \\ {\scriptstyle p_x}\uparrow & & \uparrow{\scriptstyle q_y} \\ \star_Z & \xrightarrow[\widetilde{h}_y\widetilde{f}_x\widetilde{g}_x]{} & \star_Z \end{array}$$

where the bottom and top paths come from unfolding the action of $[f + g, h]$ and $[f, h] + [g, h]$ on $\mathsf{push}\,(x, y)$.

I claim that setting $p_x = \widetilde{g}_x\widetilde{f}_x$ and $q_y = \widetilde{h}_y^{-1}$ does the job. With these choices, the square boils down to showing that

$$(\widetilde{f}_x\widetilde{g}_x)^{-1} \cdot \widetilde{h}_y\widetilde{f}_x\widetilde{g}_x \cdot \widetilde{h}_y^{-1} = \widetilde{f}_x^{-1}\widetilde{h}_y\widetilde{f}_x\widetilde{h}_y^{-1}\widetilde{g}_x^{-1}\widetilde{h}_y\widetilde{g}_x\widetilde{h}_y^{-1}$$

which, after cancelling out obvious matches on both sides, comes down to solving

$$\widetilde{g}_x^{-1}\widetilde{h}_y\widetilde{f}_x = \widetilde{h}_y\widetilde{f}_x\widetilde{h}_y^{-1}\widetilde{g}_x^{-1}\widetilde{h}_y \tag{1}$$

Here, it may seem we are stuck (and indeed I was for quite some time...) – to solve this, we would like to start swapping elements, but loop spaces are not, in general, commutative. There is however a very simple trick: we view the term $\widetilde{h}_y^{-1}\widetilde{g}_x^{-1}\widetilde{h}_y$ as a function in x. As such, it is in fact a *pointed* function $X \to_\star \Omega Z$. As we assumed X to be a suspension, it is an easy consequence of Eckmann-Hilton that for any $\alpha, \beta : X \to_\star \Omega Z$ we have that $\alpha(x)\beta(x) = \beta(x)\alpha(x)$. Hence, we may swap $\widetilde{h}_y^{-1}\widetilde{g}_x^{-1}\widetilde{h}_y$ and $\widetilde{f}_x$ in Equation (1). This cancels out the additional terms and we have the desired identity. □

All properties concerning Whitehead products listed here boil down to cute word problems like the one above and all of them can be solved by the same proof technique as the one used here, i.e. by grouping together appropriate terms, forcing Eckmann-Hilton to kick in. For the Jacobi identity especially, this trick has to be applied several times and choosing the right terms to group together is not always obvious. Fortunately for me, I do not have enough space to give the proof here.

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